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Student Name	[redacted]	Student Number	[redacted]

University of Leeds

MATH3001

Project in Mathematics
Public Engagement

Egyptian Fractions

[redacted]

[redacted]

Supervised by
[redacted] and [redacted]

March 30, 2023

Contents

1 Overview

For my public engagement project, I created a two-part animated video series that explores Egyptian fractions and their practical applications. The first video provides a general overview of the Ancient Egyptian numeration system, as well as some basic calculations. The second video focuses on the extended applications of Egyptian fractions, including additional methods and modern problems. These videos were designed for students aged 13-18 with an interest in mathematics, although the second video assumed a slightly higher level of mathematical understanding, making it also suitable for A-Level or even university level students.

In total, I received 40 responses to my videos. While it may have been tempting to falsify data to achieve favourable results, I recognise that doing so would compromise the accuracy of my conclusions and violate research standards. As such, I have presented my results honestly and transparently.

To gather feedback on my videos, I showed them to members of my target audience and asked them to fill out a feedback form provided in A. I showed my videos to four classes at a British secondary school, which will be referred to as School X throughout this report. I also showed my video to four of my maths tutees and received specific individual feedback. In order to keep their responses anonymous, I have chosen to refer to them as Tutees A, E, I and O. To protect the privacy of the participants, I collected only their age and feedback responses, without collecting any personal identifying information.

Before presenting the videos, I informed participants of the aims of my project and how their responses would be used to ensure ethical research practices. I also emphasised the confidentiality of their responses and their right to withdraw their responses at any time, obtaining informed consent from all participants. In accordance with the Information Commissioner's Office's guidance on data protection for children, I prioritised the values of privacy and anonymity when working with children.

2 Reasoning

The motivation for choosing to investigate Egyptian fractions stems from a personal interest in this topic, which initially appeared intriguing and unfamiliar. I hoped that it would also be a new and interesting topic for viewers. As my research progressed, it became increasingly apparent that Egyptian fractions are a fascinating and multifaceted area of study. The concept of unit fraction representations is particularly intriguing as it deviates from our familiar fractional numeration and presents a unique perspective on mathematical reasoning. Moreover, the topic exhibits a remarkable range of potential applications, which are both unexpected and inspiring. Notably, I believe that the connection between unit fraction representations and Sylvester's sequence is especially captivating as it unearths an unanticipated relationship between two seemingly distinct mathematical concepts. Learning about a variety of mathematical topics enhances cognitive faculties, particularly problem-solving skills, and also enriches our understanding of the fundamental concepts in mathematics.

The videos aim to demonstrate that mathematics is a language with its own alphabet, and that numbers are the symbols used to represent values. Through an analysis of the symbols and number systems used by Ancient Egyptians, it becomes clear that there are multiple ways of writing mathematics, and the conventions and symbols used are arbitrary, and developed over time. Such an understanding of mathematical notation can lead to a more flexible and innovative approach to problem-solving. Having this view of mathematics can improve lateral thinking, leading to new ideas and ways of thinking.

Furthermore, studying the history of mathematics can help to understand the development of mathematical concepts, and provide insight into their future potential. Collaborating with others and exploring their ideas about mathematics can offer a lot of learning opportunities, including drawing inspiration from different cultures and societies. In particular, the examination of Egyptian mathematics can contribute to the de-westernisation of mathematics and science, promoting open-mindedness and encouraging representation of diverse cultures and societies in STEM fields. The impact of this research is not limited to the academic realm, but can inspire the next generation of innovative mathematicians, regardless of their background or cultural heritage.

The engagement medium chosen for this is video, as it allows for the combination of visuals and audio, enhancing the accessibility and entertainment value of the project. Initially, a formal, lecture-style video was intended; however, upon further consideration, I realised it may not be as easy to produce and may not captivate the audience as effectively as an animated format. Therefore, a more informal and engaging style, characterised by cute animations, was chosen instead. This style was found to be particularly suitable for the younger demographic of the intended audience, while also being enjoyable for other age groups. Moreover, the videos offer a well-rounded and unique approach to this topic in that they do *all* of the following:

(a) Visually demonstrate the process of working out questions.

(b) Demonstrate active completion of example questions.

(c) Allow for viewer interaction and can be used to teach *how* to do Egyptian fractions not just about them.

(d) Bring together advanced mathematical problems.

(e) Discuss further applications of Egyptian fractions.

(f) Discuss historical context.

3 The Mathematics of Egyptian Fractions

In approximately 1550 BC a scribe named Ahmes copied a 200-year-old mathematical text onto a 6-metre-long scroll of papyrus [27]. Despite his lack of knowledge of the future fame his work would earn him, Ahmes went on to become the first historical mathematical figure to use fractions, as well as the earliest to be named. However, in the scroll he claims not to be the original creator of his mathematical writings, but rather a scribe, copying another document almost 500 years older!

Ahmes' scroll was lost to the sands of time, only to be rediscovered in the 17th Century in or around the Ramesseum, the memorial temple of Pharaoh Ramesses II. The scroll was sold to Alexander Henry Rhind, from whom it got its name, before being acquired by the British Museum. Along with the Egyptian Mathematical Leather Roll (EMLR) and the Akhmim wooden tablets, the Rhind Mathematical Papyrus (RMP) contains much information about the mathematics of ancient Egyptians, with a particular focus on Egyptian fractions, geometry, algebra, and series. An excellent transcription and translation can be found in [14]. However, before delving into the world of Egyptian fractions and their associated problems, it is necessary to discuss ancient Egyptian mathematical notation. Much of the historical ideas in the following sections come from 'Mathematics in Ancient Egypt: A Contextual History' [21], 'Science Awakening' [34] and 'The Secret History of Writing' [1].

3.1 Ancient Egyptian Notation

Ancient Egypt was dominated by hieroglyphics, and its mathematics was no different. Different hieroglyphic symbols, shown in Figure 3.1, were used to represent powers of 10, with the symbols functioning additively to create numbers, as seen in Figures 3.2 and 3.3.

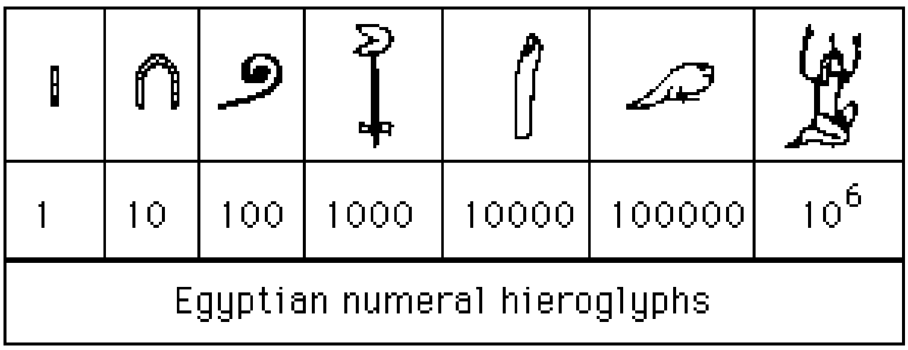

Figure 3.1: Hieroglyphic numerals [28]

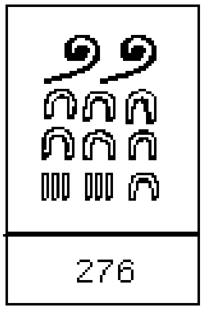

Figure 3.2: 276 in hieroglyphs [28]

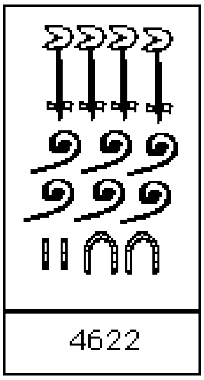

Figure 3.3: 4622 in hieroglyphs [28]

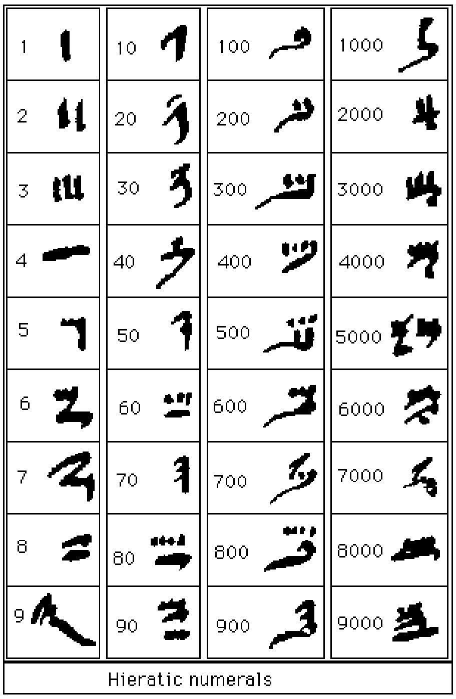

Figure 3.4: Hieratic numerals [28]

Hieratic script can be considered as to hieroglyphs what modern handwriting is to printed script. It was a faster and easier way of writing than hieroglyphs, using reed pens and papyrus instead of chisels and stone, and could be used in everyday life. As such there is a greater variation of hieratic symbols, and although clearly inspired by hieroglyphs, many symbols were transformed - arguably to make writing faster. For example, hieratic script had a symbol for each multiple of 10, not just each power of 10, which can be seen in Figure 3.4. Thus the number 2000 became one symbol, instead of two. Mathematical hieratic symbols also worked additively, and the advantage of hieratic script is

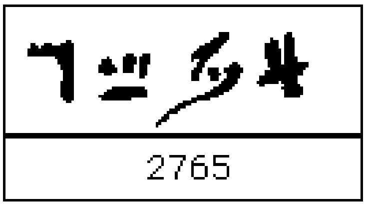

Figure 3.5: 2765 written in hieratic script, right to left [28]

Figure 3.6: 2765 written in hieratic script, left to right [28]

clear in the case of 9999, which would take 36 hieroglyphs to write, but only 4 hieratic symbols (as shown in my first video).

Despite also being base-10, ancient Egyptian symbols did not represent true place value as our modern number system does. Where our usage of columns and positions represents the value of a symbol, for hieroglyphic and hieratic script the order of symbols could be rearranged without altering the meaning - the number system was non-positional. This is clear to see in Figures 3.5 and 3.6.

Furthermore, these number systems lacked one key symbol: 0. 0 is an extremely powerful symbol and allows mathematicians to access many mathematical concepts, for example negative numbers. Additionally, without a symbol for 0 it is far more difficult to show when an equation has become balanced, or to have the solution to questions like 2 subtract 2. It has been suggested that in balanced situations, ancient Egyptian mathematicians wrote the hieroglyph *nfr* meaning ‘complete’ or ‘perfect’ shown in Figure 3.7.

Figure 3.7: *Nfr*, meaning ‘complete’ [25]

3.2 Simple Calculations

Due to this number system, ancient Egyptian problem-solving - in particular multiplication and division - was primarily based on a trial-and-error approach, often using multiples of 2 to construct a factor or multiple. These following examples are inspired by similar problems in the RMP [2, 3, 18, 22, 29, 42].

Example 1: $23 \cdot 36 = x$
Solution:
Firstly create a bank of multiples of 36. Then add these multiples to get $23 \cdot 36$.

$$\begin{aligned} 1 \cdot 36 &= 36 \\ 2 \cdot 36 &= 72 \\ 4 \cdot 36 &= 144 \\ 8 \cdot 36 &= 288 \\ 16 \cdot 36 &= 576 \end{aligned}$$

$$\begin{aligned} 23 &= 16 + 4 + 2 + 1 \\ \Rightarrow 23 \cdot 36 &= (16 + 4 + 2 + 1) \cdot 36 \\ &= 576 + 144 + 72 + 36 \\ &= 828 \end{aligned}$$

Example 2: $156 \div 12 = x$
Solution:
Firstly create a bank of multiples of 12. Then add these multiples to get $x \cdot 12 = 156$.

$$\begin{aligned} 1 \cdot 12 &= 12 \\ 2 \cdot 12 &= 24 \\ 4 \cdot 12 &= 48 \\ 8 \cdot 12 &= 96 \\ 16 \cdot 12 &= 192 \end{aligned}$$

$$\begin{aligned} 156 &= 96 + 48 + 12 \\ &= 8 \cdot 12 + 4 \cdot 12 + 1 \cdot 12 \\ &= 13 \cdot 12 \\ \Rightarrow x &= 13 \end{aligned}$$

What about cases where the answer is not a whole number? In our modern age we might use a calculator, or use our system of place value to divide and get decimal answers. Ancient Egyptian mathematicians equally recognised a need for small values less than one, and turned to fractional representations.

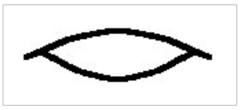

Figure 3.8: r, meaning ‘a part’ [23]

3.3 Fractions

In many ways ancient Egyptian fractions worked just as our modern ones do. They had a symbol denoting fractional nature, which was usually written over, or to the side of, the numbers of the denominator, just like $\frac{1}{2}$ and 1/2. However, all the fractions (bar a couple of exceptions shown in Figure 3.9) were unit fractions, written in the form $\frac{1}{n}$. The symbol denoting ‘one over’ was the hieroglyphic symbol r, representing a mouth, and meaning ‘a part’, shown in Figure 3.8. Thus it can be understood that ancient Egyptian fractions were written as ‘a part out of n parts’.

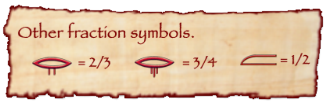

Figure 3.9: Hieroglyphic exceptions to the 1/n fraction format [24]

Example 3: Write $\frac{1}{5}$ and $\frac{1}{3}$ in Egyptian hieroglyph form.
Solution:

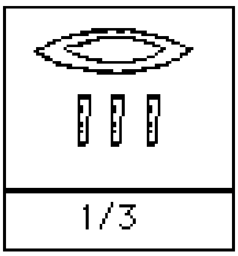

Figure 3.10: 1/3 [28]

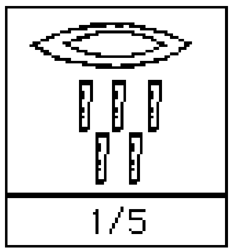

Figure 3.11: 1/5 [28]

The hieratic form of fractions followed the same rules, but were written as the hieratic symbol with a dot over it (instead of the hieroglyph for ‘a part’).

This is simple to construct for fractions already of the form $\frac{1}{n}$, but for other fractions with larger denominators it can be much more difficult. Ancient Egyptians wrote such fractions as sums of distinct unit fractions. The RMP contains a 2/n table, spread over nine sheets of papyrus, which describes the Egyptian fraction decompositions of some fifty fractions of the form 2/n, n odd [22, 39, 42].

Example 4:

$$\frac{2}{47} = \frac{1}{30} + \frac{1}{41} + \frac{1}{470}$$

$$\frac{2}{91} = \frac{1}{70} + \frac{1}{130}$$

It is from this that we gain our modern definition of an Egyptian fraction - **a finite sum of distinct unit fractions**.

Figure 3.12: The Rhind Mathematical Papyrus [39]

2/3 = 1/2 + 1/6	2/5 = 1/3 + 1/15	2/7 = 1/4 + 1/28
2/9 = 1/6 + 1/18	2/11 = 1/6 + 1/66	2/13 = 1/8 + 1/52 + 1/104
2/15 = 1/10 + 1/30	2/17 = 1/12 + 1/51 + 1/68	2/19 = 1/12 + 1/76 + 1/114
2/21 = 1/14 + 1/42	2/23 = 1/12 + 1/276	2/25 = 1/15 + 1/75
2/27 = 1/18 + 1/54	2/29 = 1/24 + 1/58 + 1/174 + 1/232	2/31 = 1/20 + 1/124 + 1/155
2/33 = 1/22 + 1/66	2/35 = 1/30 + 1/42	2/37 = 1/24 + 1/111 + 1/296
2/39 = 1/26 + 1/78	2/41 = 1/24 + 1/246 + 1/328	2/43 = 1/42 + 1/86 + 1/129 + 1/301
2/45 = 1/30 + 1/90	2/47 = 1/30 + 1/141 + 1/470	2/49 = 1/28 + 1/196
2/51 = 1/34 + 1/102	2/53 = 1/30 + 1/318 + 1/795	2/55 = 1/30 + 1/330
2/57 = 1/38 + 1/114	2/59 = 1/36 + 1/236 + 1/531	2/61 = 1/40 + 1/244 + 1/488 + 1/610
2/63 = 1/42 + 1/126	2/65 = 1/39 + 1/195	2/67 = 1/40 + 1/335 + 1/536
2/69 = 1/46 + 1/138	2/71 = 1/40 + 1/568 + 1/710	2/73 = 1/60 + 1/219 + 1/292 + 1/365
2/75 = 1/50 + 1/150	2/77 = 1/44 + 1/308	2/79 = 1/60 + 1/237 + 1/316 + 1/790
2/81 = 1/54 + 1/162	2/83 = 1/60 + 1/332 + 1/415 + 1/498	2/85 = 1/51 + 1/255
2/87 = 1/58 + 1/174	2/89 = 1/60 + 1/356 + 1/534 + 1/890	2/91 = 1/70 + 1/130
2/93 = 1/62 + 1/186	2/95 = 1/60 + 1/380 + 1/570	2/97 = 1/56 + 1/679 + 1/776
2/99 = 1/66 + 1/198	2/101 = 1/101 + 1/202 + 1/303 + 1/606	

Figure 3.13: 2/n table from the RMP [39]

3.4 The Greedy Algorithm

There are many ways to construct Egyptian fractions, one such way being Fibonacci's Greedy Algorithm [20, 22].

Fibonacci's Greedy Algorithm

- Given a rational fraction of the form $\frac{a}{b}$, if a = 1 then the algorithm terminates. Otherwise, subtract the greatest fraction of the form $\frac{1}{n}$, $n \in \mathbb{R}$, which gives a non-negative remainder.
- The solution to this subtraction is of the form $\frac{a'}{b'}$.
- Using $\frac{a'}{b'}$ as start value, repeat the algorithm.
- Repeat until algorithm terminates.

The Egyptian fraction form of $\frac{a}{b}$ is the sum of the fractions of the form $\frac{1}{n}$.

Example 5: Write $\frac{7}{12}$ as an Egyptian fraction using Fibonacci's Greedy Algorithm.
Solution:

$$\begin{aligned} \frac{7}{12} - \frac{1}{12} &= \frac{1}{2} \\ \Rightarrow \frac{7}{12} &= \frac{1}{12} + \frac{1}{2} \end{aligned}$$

The algorithm can also be expressed using the ceiling function.

$$\frac{a}{b} = \frac{1}{\left\lceil \frac{b}{a} \right\rceil} + \frac{(-b) \bmod a}{b \left\lceil \frac{b}{a} \right\rceil} \tag{1}$$

Example 6: Write $\frac{11}{12}$ as an Egyptian fraction using (1).
Solution:

$$\begin{aligned} \frac{11}{12} &= \frac{1}{2} + \frac{5}{12} \\ \frac{5}{12} &= \frac{1}{3} + \frac{1}{12} \\ \Rightarrow \frac{11}{12} &= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} \end{aligned}$$

Evidently this greedy algorithm cannot be used to create expansions which use the Egyptian exception symbols like 2/3 etc. To allow for such representations I have modified the algorithm.

Modified Greedy Algorithm

- Given a rational fraction of the form $\frac{a}{b}$, if a = 1 then the algorithm terminates. Otherwise, subtract the greatest fraction $\frac{p}{q}$ which gives a non-negative remainder, **where $\mathbf{\frac{p}{q}}$ can be of the form i. $\mathbf{\frac{2}{3}}$, ii. $\mathbf{\frac{3}{4}}$ or iii. $\mathbf{\frac{1}{n}}$, n** $\in \mathbb{R}$.

- The solution to this subtraction is of the form $\frac{a'}{b'}$.

- Using $\frac{a'}{b'}$ as start value, repeat the algorithm.

- Repeat until algorithm terminates.

The Egyptian fraction form of $\frac{a}{b}$ is the sum of the fractions subtracted in the algorithm.

Example 7: Write $\frac{8}{10}$ as an Egyptian fraction using a. the Greedy Algorithm and b. the Modified Greedy Algorithm.
Solution:
a.

$$\frac{8}{10} - \frac{1}{2} = \frac{3}{10}$$
$$\frac{3}{10} - \frac{1}{5} = \frac{1}{10}$$
$$\Rightarrow \frac{8}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$$

b.

$$\frac{8}{10} - \frac{2}{3} = \frac{2}{15}$$
$$\frac{2}{15} - \frac{1}{10} = \frac{1}{30}$$
$$\Rightarrow \frac{8}{10} = \frac{2}{3} + \frac{1}{10} + \frac{1}{30}$$

3.5 The Splitting Algorithm

Another method for creating Egyptian fractions is the splitting algorithm [4,22], for which proofs can be found in [4,6]. This algorithm uses the Splitting Identity (2).

$$\frac{1}{x_k} = \frac{1}{x_k+1} + \frac{1}{x_k(x_k+1)} \tag{2}$$

Splitting Algorithm

- Given a sum of unit fractions $\frac{1}{x_1}+\ldots+\frac{1}{x_L}$, where $x_1,\ldots,x_L \in \mathbb{Z}^+$, if any x_k has a multiplicity $\mu > 1$, then $(\mu-1)$ copies of $\frac{1}{x_k}$ are replaced using (2).

- Repeat until there are no more duplicate unit fractions.

For a rational fraction of the form $\frac{a}{b}$, let $\frac{a}{b}$ be written as the sum of a unit fractions of the form $\frac{1}{b}$. Then apply the splitting algorithm.

Example 8: Write $\frac{3}{11}$ as an Egyptian fraction using the Splitting Algorithm.
Solution:

$$\begin{aligned}\frac{3}{11} &= \frac{1}{11}+\frac{1}{11}+\frac{1}{11}\\ &= \frac{1}{11}+\left(\frac{1}{12}+\frac{1}{132}\right)+\left(\frac{1}{12}+\frac{1}{132}\right) \text{ by applying (2).}\\ &= \frac{1}{11}+\frac{1}{12}+\frac{1}{132}+\frac{1}{13}+\frac{1}{156}+\frac{1}{133}+\frac{1}{17556}\end{aligned}$$

3.6 Farey Sequences

Given two fractions in simplest form (irreducible fractions), $\frac{a}{b}$ and $\frac{c}{d}$, adding the numerators and the denominators gives a fraction $\frac{a+c}{b+d}$ that falls between $\frac{a}{b}$ and $\frac{c}{d}$. This fraction is known as a mediant and may not always be exactly halfway between $\frac{a}{b}$ and $\frac{c}{d}$ [20,26].

Theorem 3.1. *If $\frac{a}{b} < \frac{c}{d}$ then the mediant is $\frac{a+c}{b+d}$, such that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Proof.

$$\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc \tag{3}$$

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{ab+bc-ab-ad}{b(b+d)} = \frac{bc-ad}{b(b+d)} > 0 \text{ by } (3)$$
$$\Rightarrow \frac{a}{b} < \frac{a+c}{b+d}$$

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc+cd-ad-cd}{d(b+d)} = \frac{bc-ad}{d(b+d)} > 0 \text{ by } (3)$$
$$\Rightarrow \frac{c}{d} > \frac{a+c}{b+d}$$

Hence, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

The Farey sequence F_n for any positive integer n is the set of irreducible rational numbers $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $\gcd(a,b) = 1$ arranged in increasing order [20]. Equivalently $F_n = \{\frac{a}{b}|\ 0 \leq \frac{a}{b} \leq 1, \gcd(a,b) = 1, n \geq b\}$. This sequence may be used to find Egyptian fractions.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

Farey neighbours are two fractions $\frac{a}{b} < \frac{c}{d}$ that are adjacent in a given Farey sequence, F_n. They have the property that $|bc - ad| = 1$.

Theorem 3.2. *For $\frac{a}{b}$ and $\frac{c}{d}$ with $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$, these fractions are Farey neighbours iff $|bc - ad| = 1$.*

Proof. For $\frac{a}{b}$ and $\frac{c}{d}$ are Farey neighbours, we can prove by induction that $|bc - ad| = 1$.
Base Case: $F_1 = \{\frac{0}{1}, \frac{1}{1}\}$ so $|bc - ad| = |1 - 0| = 1$ hence this holds for n=1.
Assume it holds for $F_n = \{..., \frac{a}{b}, \frac{c}{d}, ...\}$ i.e. $|bc - ad| = 1$.
Then $F_{n+1} = \{..., \frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}, ...\}$, and $|b(a+c) - a(b+d)| = |ab + bc - ab - ad| = |bc - ad| = 1$.
Hence this holds for $n+1$, and so by the principal of induction, if $\frac{a}{b}$ and $\frac{c}{d}$ are Farey neighbours, then $|bc - ad| = 1$ for all $n \in \mathbb{Z}^+$.

For the converse, if we have three values in some Farey sequence, $\frac{a}{b}, \frac{x}{y}, \frac{c}{d}$

such that $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$ and $bx - ay = cy - dx = 1$ then we can see that

$$\begin{aligned} bx + dx &= cy + ay \\ \Rightarrow x(b+d) &= y(a+c) \\ \Rightarrow \frac{x}{y} &= \frac{a+c}{b+d}. \end{aligned}$$

Hence $\frac{x}{y}$ is the mediant between $\frac{a}{b}$ and $\frac{c}{d}$ so for some Farey sequence they are neighbours (although for later Farey sequences subsequent mediants may separate them).

□

To construct Egyptian fractions using the Farey sequence we can use the algorithm developed by Bleicher in 1968 [5, 15]. It is useful to first find a relationship between elements in a given Farey sequence F_{b_n}.

Proposition 3.1. *For Farey neighbours in a Farey sequence* F_{b_n}, $f_n = f_{n+1} + \frac{1}{b_n b_{n+1}}$.

Proof. Begin by defining the sequence such that $\frac{a}{b} = f_{n+1} = \frac{a_{n+1}}{b_{n+1}} < f_n$ is the neighbouring term to $\frac{c}{d} = f_n = \frac{a_n}{b_n}$. We know from Theorem 3.2 that for Farey neighbours $bc - ad = 1$.
Then,

$$\begin{aligned} &bc = 1 + ad \\ &\Rightarrow c = \frac{1+ad}{b} \\ &\Rightarrow \frac{c}{d} = \frac{1+ad}{bd} \\ &\Rightarrow \frac{c}{d} = \frac{a}{b} + \frac{1}{bd} \\ &\Rightarrow \frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}} + \frac{1}{b_n b_{n+1}} \\ &\Rightarrow f_n = f_{n+1} + \frac{1}{b_n b_{n+1}} \end{aligned} \tag{4}$$

□

Farey Sequence Algorithm

- Given a rational, non-zero, irreducible fraction $f_n = \frac{a_n}{b_n}$ in a Farey sequence F_{b_n}, the next smallest element is the adjacent term $f_{n+1} < f_n$. Subtract f_{n+1} from f_n. (Recall from (4) that $f_n = f_{n+1} + \frac{1}{b_n b_{n+1}}$, so the result of this subtraction will itself be a unit fraction: $\frac{1}{b_n b_{n+1}}$).

- Repeat until $f_{n+1} = 0$ i.e. all unit fraction differences have been found.

f_n can then be expressed as the sum of the unit fraction differences (the solutions to the subtractions).

It is clear that as $b_n \leq n$ we must have that $b_{n+1} \leq b_n$. Then as $f_{n+1} < f_n$ we must have that $a_{n+1} < a_n$. Thus the sequence for a_n is strictly decreasing and must eventually reach 0, hence Farey sequence Egyptian fractions must be finite.

Example 9: Write $\frac{4}{5}$ as an Egyptian fraction using the Farey Sequence Algorithm.
Solution:
Begin with the Farey sequence containing $\frac{4}{5}$.
$F_5 = \{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\}$

$$\frac{4}{5} - \frac{3}{4} = \frac{1}{20}$$
$$\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$
$$\frac{2}{3} - \frac{3}{5} = \frac{1}{15}$$
$$\frac{3}{5} - \frac{1}{2} = \frac{1}{10}$$
$$\frac{1}{2} \text{ has numerator 1, hence the algorithm terminates.}$$
$$\Rightarrow \frac{4}{5} = \frac{1}{20} + \frac{1}{12} + \frac{1}{15} + \frac{1}{10} + \frac{1}{2}$$

3.7 Practical Numbers

Many of the ideas for this section can be found in [33, 35, 38]. A proper divisor of a number n is a divisor of n, excluding n itself. A practical number, N, is a positive integer such that all positive integers less than N can be written as the sum of its proper divisors. Examples include 1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36,...

Practical numbers can be used to write Egyptian fractions. Given a fraction $\frac{a}{b}$ with b being a practical number and $a < b$, we can then write a as a sum of the proper divisors of b. Thus we can write $\frac{a}{b}$ as a sum $\sum \frac{d_i}{b}$ with d_i being the proper divisors of b. These fractions can then be simplified to unit fractions as we can divide both the top and bottom of each fraction by d_i, giving that $\frac{a}{b} = \sum \frac{1}{b/d_i}$.

Example 10: Show 12 is a practical number.
Solution:
The proper divisors of 12 are 1,2,3,4,6. The positive integers less than 12 are 1,2,3,4,5,6,7,8,9,10,11.

$$\begin{aligned} 1 &= 1 & 7 &= 6+1 \\ 2 &= 2 & 8 &= 6+2 \\ 3 &= 3 & 9 &= 6+3 \\ 4 &= 4 & 10 &= 6+4 \\ 5 &= 1+4 & 11 &= 6+4+1 \\ 6 &= 6 \end{aligned}$$

All the positive integers less than 12 can be written as a sum of its proper divisors, thus 12 is a practical number.

Example 11: Write 7/18 as an Egyptian fraction using practical numbers.
Solution:
The proper divisors of 18 are 1, 2, 3, 6, 9.

$$\begin{aligned} 7 &= 6+1 \\ \Rightarrow \frac{7}{18} &= \frac{1}{18} + \frac{6}{18} \\ &= \frac{1}{18} + \frac{1}{3} \end{aligned}$$

3.8 Other Methods

There are many more methods which can be used to write Egyptian fractions, some of which are described below [12, 36].

For fractions of the form $\frac{n}{pq}$, where $p+q$ is a multiple of n,

$$\frac{n}{pq} = \frac{n}{p(p+q)} + \frac{n}{q(p+q)} \tag{5}$$

$$= \frac{1}{\frac{p(p+q)}{n}} + \frac{1}{\frac{q(p+q)}{n}} \tag{6}$$

For a fraction of the form $\frac{2}{n}$ there are several simple expansions which can be used to form Egyptian fractions.

- If $n = 0 \mod 3$:

$$\frac{2}{n} = \frac{1}{2} \cdot \frac{1}{n} + \frac{2}{3} \cdot \frac{1}{n} \tag{7}$$

- If $n = 0 \mod 5$:

$$\frac{2}{n} = \frac{1}{3} \cdot \frac{1}{n} + \frac{5}{3} \cdot \frac{1}{n} \tag{8}$$

- In a more general case, we have

$$\frac{2}{n} = \frac{2}{n+1} + \frac{2}{n(n+1)} \tag{9}$$
$$= \frac{1}{\frac{n+1}{2}} + \frac{1}{\frac{n(n+1)}{2}} \tag{10}$$

- For fractions of the form $\frac{2}{p}$, with p prime, where $\frac{p}{2} < A < p$, and A has many divisors:

$$\frac{2}{p} = \frac{1}{A} + \frac{2A - p}{A \cdot p} \tag{11}$$

Alternatively, we may have

$$\frac{2}{p} = \frac{1}{p} + \frac{1}{2p} + \frac{1}{3p} + \frac{1}{6p} \tag{12}$$

This small selection of possible expansion equations demonstrates how varied and easy to manipulate the Egyptian fractions are, which is what lends them to being very useful in many different applications - sometimes even more useful than modern fractions. For example, consider splitting loaves of bread or (as in my video!) cakes, between multiple people. A modern person might turn to their mobile phone, or calculator, and split the number of loaves (n) over the number of people (q), giving $\frac{n}{q}$, then split each loaf into q parts and give each person n of them. However, this is not necessarily the best or most efficient method - it involves all the loaves being split into many parts, and each person will get many small pieces. An Egyptian fraction expansion, however, may be far simpler, as seen in the example below which comes from the Rhind Papyrus.

Example 12: Share 6 loaves of bread between 10 people
Solution:
Write this as the fraction $\frac{6}{10}$ then expand this in Egyptian form.
Using the Greedy Algorithm

$$\frac{6}{10} - \frac{1}{2} = \frac{1}{10}$$
$$\Rightarrow \frac{6}{10} = \frac{1}{2} + \frac{1}{10}$$

Thus each guest receives $\frac{1}{2}$ and $\frac{1}{10}$ of a cake.This is clearly much easier to do than split each loaf into 10, and give each person 6 pieces.

3.9 Comparing Methods

The efficiency of writing out $\frac{x}{y}$ in terms of the number of terms can be expressed for different algorithms.

First, using practical numbers, the number of terms is O(log(y)), meaning that it grows slowly as y increases. This suggests that this approach is efficient and can be applied to large denominators without excessive computational cost.

For the greedy algorithm, the maximum number of terms in the representation of $\frac{x}{y}$ is x - the numerator decreases after each stage of the algorithm. This approach is also efficient and can be applied to large numerators.

Using the splitting algorithm, if no number appears twice in the splitting identity i.e. $\frac{1}{y_n+1} \neq \frac{1}{y_{n+1}(y_{n+1}+1)} \forall n$, we can consider the algorithm as having $x-1$ splits. Each step of the algorithm doubles the number of terms in the representation, so the total number of terms is O(x) In practice, however, numbers do sometimes appear twice in the splitting algorithm so the total number of terms could be much larger, and of a greater order.

Overall, it can be seen that the practical approach is efficient for large denominators and that the greedy approach is efficient for large numerators. The splitting algorithm can also be used but may be less efficient in practice.

4 Modern Problems and Applications

4.1 Engel Expansions

Some of the ideas in this section come from [13, 37]. Engel's expansion states that given $x \in \mathbb{R}^+$, $x = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \ldots$, where $(a_1, a_2, a_3, \ldots)$ is a unique non-decreasing sequence of positive integers. For irrational x the Engel expansion is infinite, for rational x the Engel expansion is finite and is an Egyptian fraction expansion for x, which I prove below.

Engel Expansion Algorithm for rational x
Given a rational, positive number x, let

$$x = u_1$$
$$a_i = \left\lceil \frac{1}{u_i} \right\rceil$$
$$u_{i+1} = (u_i \cdot a_i) - 1$$

Repeat until $u_i = 0$

Theorem 4.1. *Rational numbers have finite Engel expansions.*

Proof. Let $\frac{x}{y} \in \mathbb{Q}$ with $y \neq 0$ i.e. $x, y \in \mathbb{Z}$.
Follow the Engel Expansion Algorithm with $u_1 = \frac{x}{y}$ and $a_1 = \lceil \frac{y}{x} \rceil$:

$$u_2 = (u_1 \cdot a_1) - 1 = \left(\frac{x}{y} \cdot \left\lceil \frac{y}{x} \right\rceil\right) - 1 = \frac{x \cdot \lceil \frac{y}{x} \rceil - y}{y}$$

By the definition of the ceiling function,

$$\frac{y}{x} \leq \left\lceil \frac{y}{x} \right\rceil < \frac{y}{x} + 1$$
$$\Rightarrow \frac{y}{x} - 1 \leq \left\lceil \frac{y}{x} \right\rceil - 1 < \frac{y}{x}$$
$$\Rightarrow \left\lceil \frac{y}{x} \right\rceil - 1 < \frac{y}{x} \leq \left\lceil \frac{y}{x} \right\rceil$$

Multiplying by x gives

$$x \cdot \left\lceil \frac{y}{x} \right\rceil - x < y$$

Adding x and subtracting y gives

$$x \cdot \left\lceil \frac{y}{x} \right\rceil - y < x \tag{13}$$

The RHS of (13), x, is the numerator for u_1 whilst the LHS, $x \cdot \lceil \frac{y}{x} \rceil - y$, is the numerator for u_2. Thus the numerator for $u_2 <$ numerator for u_1. This holds for all u_i (the proof by induction of which is trivial) so for all i the numerator of $u_{i+1} <$ the numerator of u_i. Clearly these numerators are strictly decreasing and so will eventually reach 0, at which point the algorithm stops. Thus the Engel expansion for a rational number is finite.

□

Example 13: Write $\frac{49}{40}$ as an Egyptian fraction using the Engel Expansion.
Solution:

$$u_1 = \frac{49}{40} \qquad a_1 = \left\lceil \frac{40}{49} \right\rceil = 1$$

$$u_2 = \frac{49}{40} \cdot 1 - 1 = \frac{9}{40} \qquad a_2 = \left\lceil \frac{40}{9} \right\rceil = 5$$

$$u_3 = \frac{9}{40} \cdot 5 - 1 = \frac{1}{8} \qquad a_3 = \left\lceil \frac{8}{1} \right\rceil = 8$$

$$u_4 = \frac{1}{8} \cdot 8 - 1 = 0$$

$$\begin{aligned} \text{Thus } \frac{49}{40} &= \frac{1}{1} + \frac{1}{1 \cdot 5} + \frac{1}{1 \cdot 5 \cdot 8} \\ &= 1 + \frac{1}{5} + \frac{1}{40} \end{aligned}$$

4.2 Sylvester's Sequence

Sylvester's sequence is an integer sequence for which each term is the product of the previous terms, plus 1. It can be used to find a finite representation of 1 in Egyptian fractions. Ideas for this section can be found in [13, 16, 40].

The first 6 terms are 2,3,7,43,1807,3263443,. . ..The recursive relationship describing these terms is

$$s_j = 1 + \prod_{i=0}^{j-1} s_i \tag{14}$$

which can also be written as

$$\begin{aligned} s_j &= 1 + \prod_{i=0}^{j-1} s_i \\ &= 1 + s_{j-1} \prod_{i=0}^{j-2} s_i \end{aligned}$$

Using the above relationship (14) we can see that

$$s_j = 1 + s_{j-1}\left(s_{j-1} - 1\right) = {s_{j-1}}^2 - s_{j-1} + 1 \tag{15}$$

The sum of the reciprocals of Sylvester's sequence is

$$\boxed{\sum_{i=0}^{\infty} \frac{1}{s_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots} \tag{16}$$

The first three partial sums are $u_1 = \frac{1}{2}$, $u_2 = \frac{5}{6}$, $u_3 = \frac{41}{42}$ and the general form of the partial sums (which I have proved by induction below) is

$$\boxed{u_j = \sum_{i=0}^{j-1} \frac{1}{s_i} = 1 - \frac{1}{s_j - 1} = \frac{s_j - 2}{s_j - 1}} \tag{17}$$

Theorem 4.2. *The general form of the partial sums for reciprocals of Sylvester's sequence is* $u_j = \sum_{i=0}^{j-1} \frac{1}{s_i} = 1 - \frac{1}{s_j - 1} = \frac{s_j - 2}{s_j - 1}$.

Proof. Base Case: Let $j = 1$.

$$\text{a.} \sum_{i=0}^{j-1} \frac{1}{s_i} = \sum_{i=0}^{0} \frac{1}{s_i} = s_0 = \frac{1}{2}, \qquad \text{b.} \frac{s_j - 2}{s_j - 1} = \frac{s_1 - 2}{s_1 - 1} = \frac{3 - 2}{3 - 1} = \frac{1}{2}$$

Inductive Hypothesis:
Assume that

$$u_j = \sum_{i=0}^{j-1} \frac{1}{s_i} = \frac{s_j - 2}{s_j - 1}.$$

Then,

$$\begin{aligned} u_{j+1} &= \sum_{i=0}^{j} \frac{1}{s_i} \\ &= \sum_{i=0}^{j-1} \frac{1}{s_i} + \frac{1}{s_j} \\ &= \frac{s_j - 2}{s_j - 1} + \frac{1}{s_j}, \text{ by the inductive hypothesis.} \\ &= \frac{s_j(s_j - 2) + s_j - 1}{s_j(s_j - 1)} \\ &= \frac{{s_j}^2 - s_j - 1}{{s_j}^2 - s_j} \\ &= \frac{s_{j+1} - 2}{s_{j+1} - 1} \text{ by (15).} \end{aligned}$$

□

From this it is clear that the sum of the reciprocals converges to 1, which I briefly detail below.

$$\sum_{i=0}^{\infty} \frac{1}{s_i} = \lim_{j\to\infty} u_j$$
$$= \lim_{j\to\infty} \frac{s_j - 2}{s_j - 1}$$
$$= 1$$

The series of the sum of the reciprocals therefore gives an infinite Egyptian fraction representation for 1.

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots$$

Thus by truncating the series at any point and subtracting 1 from the final denominator, we can find a finite Egyptian representation for 1.

Sylvester Algorithm
For the sum of the inverses of Sylvester's sequence, $\sum_{i=0}^{\infty} \frac{1}{s_i}$, to find an Egyptian fraction representation of 1 of length n, truncate the sum at the n$^{\text{th}}$ term $\frac{1}{s_n}$ and replace $\frac{1}{s_n}$ with $\frac{1}{s_n - 1}$.

We can intuitively understand how Sylvester's algorithm works by considering it as a greedy algorithm, where at each point in the algorithm we choose the smallest denominator to make the sum underestimate 1.

Example 14: Using Sylvester's sequence, find a finite Egyptian fraction expansion of 1, using a. 3 fractions and b. 5 fractions.
Solution:
a. For n = 3

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7-1}$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

b. For n = 5

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807-1}$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806}$$

Furthermore, the first k terms of the infinite series give the closest underestimate of 1 for any k-term Egyptian fraction representation e.g. $k = 4$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} = \frac{1805}{1806}$. For any number in the open interval $(\frac{1805}{1806}, 1)$ the Egyptian fraction representation has at least 5 terms. I have proved this below [30, 32].

First let us consider Sylvester's sequence, s_i, where $s_j = s_0 s_1 s_2 \cdots s_{j-1} + 1$. We have already proven that the general form of each partial sum is (17)

$$u_j = \sum_{i=0}^{j-1} \frac{1}{s_i} = 1 - \frac{1}{s_j - 1}$$

Then we can say that

$$\begin{aligned} u_{j+1} &= u_j + \frac{1}{s_j} && (18) \\ &= 1 + \frac{-s_j + s_j - 1}{s_j (s_j - 1)} && (19) \\ &= 1 - \frac{1}{s_j (s_j - 1)} && (20) \\ & && (21) \end{aligned}$$

Considering the definition of s_j, we can see that

$$\begin{aligned} s_j - 1 &= s_0 s_1 s_2 \cdots s_{j-1} \\ s_j (s_j - 1) &= s_j (s_0 s_1 s_2 \cdots s_{j-1}) \\ &= s_{j+1} - 1 \end{aligned}$$

$$\Rightarrow u_{j+1} = 1 - \frac{1}{s_{j+1} - 1}$$

Proposition 4.1. *Let $v > s_j$, then $u_j + \frac{1}{v} < u_{j+1}$*

Proof.

$$\begin{aligned} u_j + \frac{1}{v} &= 1 - \frac{1}{s_j - 1} + \frac{1}{v} \\ &< 1 + \frac{-s_j + s_j - 1}{s_j (s_j - 1)} \\ &= u_{j+1} \text{ by } (20) \\ \Rightarrow & u_j + \frac{1}{v} < u_{j+1} \end{aligned}$$

□

Proposition 4.2. *Let $u \leq s_j - 1$, then $u_j + \frac{1}{u} \geq 1$*

Proof.

$$\begin{aligned} u_j + \frac{1}{u} &= 1 - \frac{1}{s_j - 1} + \frac{1}{u} \\ &\geq 1 - \frac{1}{u} + \frac{1}{u} \\ \Rightarrow & u_j + \frac{1}{u} \geq 1 \end{aligned}$$

□

Then using these results it is clear that if u_j is the best k-term unit fraction approximation to 1 (where $j = k - 1$), then u_{j+1} is the best $(k+1)$-term unit fraction approximation to 1. $\square$

It is important to note that this proof depends upon the fact that each u_{j+1} must include the previous k fractions from u_j.

4.3 Primary Pseudoperfect Numbers

A pseudoperfect number is a positive integer which is the sum of some or all of its proper divisors e.g. $20 = 1 + 4 + 5 + 10$. A pseudoperfect number which is the sum of *all* its proper divisors is a perfect number. These numbers feature in some interesting mathematical problems, and I will use them in the next section to find solutions of such a problem. Some of the mathematics described in this section can be found in [11, 31, 35].

A primary pseudoperfect number (PPN) is an integer $N > 1$ such that the reciprocals of it and its prime factors sum to 1. There are eight known primary pseudoperfect numbers:

$2, 6, 42, 1806, 47058, 2214502422, 52495396602, 8490421583559688410706771261086$

Interestingly, the first four of these are one less than the first four terms of Sylvester's sequence.

Primary Pseudoperfect Algorithm
Let $p|N$ be the sequence of prime divisors of N, then N is a PPN if

$$\frac{1}{N} + \sum_{p|N} \frac{1}{p} = 1 \tag{22}$$

Equivalently, N is a PPN if

$$1 + \sum_{p|N} \frac{N}{p} = N \tag{23}$$

PPNs can be used to help find solutions to modern problems, like Znàm's problem.

4.4 Znàm's Problem

Znàm's problem [8, 10, 41] searches for sets of integers where each element is a proper divisor of the product of the other elements, plus 1. Alternatively, it asks which sets $\{n_1, \ldots, n_k\} \in \mathbb{Z}$ exist such that for each j, n_j divides (but is not equal to)

$$\left(\prod_{i \neq j}^{k} n_i\right) + 1 \tag{24}$$

Example 15: One such sequence for $k = 5$ is {2,3,7,47,395}.

$$3 \cdot 7 \cdot 47 \cdot 395 + 1 = 389866 = 0 \bmod 2, \neq 2$$
$$2 \cdot 7 \cdot 47 \cdot 395 + 1 = 259911 = 0 \bmod 3, \neq 3$$
$$2 \cdot 3 \cdot 47 \cdot 395 + 1 = 111391 = 0 \bmod 7, \neq 7$$
$$2 \cdot 3 \cdot 7 \cdot 395 + 1 = 16591 = 0 \bmod 47, \neq 47$$
$$2 \cdot 3 \cdot 7 \cdot 47 + 1 = 1975 = 0 \bmod 395, \neq 395$$

Znàm's improper problem asks which sets of integers have the property that each element is a divisor (not necessarily a proper divisor) of the product of the other elements, plus 1. Thus any set of integers which is a solution to Znàm's problem is also a solution to Znàm's improper problem.

Division by the product of the values n_j shows that solutions to Znàm's improper problem are solutions to

$$\sum_{j=1}^{k} \frac{1}{n_j} + \prod_{j=1}^{k} \frac{1}{n_j} = m \tag{25}$$

with $m, n_j \in \mathbb{Z}$. I have detailed the process below.

Theorem 4.3. *Solutions to Znàm's improper problem, i.e. divisors of* $\left(\prod_{i \neq j}^{k} n_i\right) + 1$, *are solutions to* (25).

Proof. Let n_j be a divisor of $\left(\prod_{i \neq j}^{k} n_j\right) + 1$.
This is equivalent to saying

$$\left(\prod_{i \neq j}^{k} n_j\right) = -1 \mod n_j$$

Then, for all j,

$$\sum_{j=1}^{k} \prod_{i \neq j}^{k} n_i + 1 = 0 \mod n_j$$

This can be considered as

$$\sum_{j=1}^{k} \prod_{i \neq j} n_i + 1 = m \cdot \prod_{j=1}^{k} n_j$$

where $m \in \mathbb{Z}^+$. If we then divide by the product $\prod_{j=1}^{k} n_j$ we get

$$\frac{\sum_{j=1}^{k} \prod_{i \neq j} n_i}{\prod_{j=1}^{k} n_j} + \frac{1}{\prod_{j=1}^{k} n_j} = \frac{m \cdot \prod_{j=1}^{k} n_j}{\prod_{j=1}^{k} n_j}$$

Which becomes

$$\frac{\sum_{j=1}^{k}\prod_{i\neq j} n_i}{\prod_{j=1}^{k} n_j} + \prod_{j=1}^{k}\frac{1}{n_j} = m \tag{26}$$

Let us consider the first term of (26). It is the sum of the products of n_i excluding the n_j term, divided by the product of *all* the n_j terms. This would simplify to a fraction $\frac{1}{n_j}$, most easily seen in an example e.g. for k=3:

$$\frac{\sum_{j=1}^{3}\prod_{i\neq j} n_i}{\prod_{j=1}^{3} n_j} = \frac{(n_2 \cdot n_3) + (n_1 \cdot n_3) + (n_1 \cdot n_2)}{n_1 \cdot n_2 \cdot n_3} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$$

Overall we can write $\sum \frac{1}{n_j}$, and thus we retrieve (25):

$$\sum_{j=1}^{k}\frac{1}{n_j} + \prod_{j=1}^{k}\frac{1}{n_j} = m$$

□

All known solutions are solutions to (25) with $m = 1$,

$$\sum_{j=1}^{k}\frac{1}{n_j} + \prod_{j=1}^{k}\frac{1}{n_j} = 1 \tag{27}$$

leading to Egyptian fraction representations of 1 as a sum of unit fractions.

4.5 Erdős-Straus and Sierpiński Conjectures

Much of the mathematics in this section comes from [7, 9, 12, 17, 19, 22].

There are some problems that are inspired by, or based upon, Egyptian fractions that have yet to be solved! For example, does a rule exist that allows us to write every fraction as the sum of three unit fractions? It has been proven that every fraction $\frac{3}{n}$ where n is not a multiple of 3, and n is odd, can be written as $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ for distinct, odd, a,b,c. The Erdős-Straus conjecture is that we can find a similar rule for fractions $\frac{4}{n}$.

Erdős-Straus Conjecture
Every fraction $\frac{4}{n}$, with $n \geq 2$, can be written as the sum of three unit fractions.

This has been verified up to the 51000000^{th} prime, $n = 1003162753$, but has not been proven or disproven. Sierpiński suggested this is also true for fractions $\frac{5}{n}$, which has been verified up to $n = 922321$.

Example 16: Some examples that supports the Erdős-Straus conjecture are

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{26} + \frac{1}{52}$$
$$\frac{4}{19} = \frac{1}{6} + \frac{1}{38} + \frac{1}{57}$$
$$\frac{4}{42} = \frac{1}{12} + \frac{1}{126} + \frac{1}{252}$$

5 Results

5.1 Survey

As part of my research project, I conducted a series of video presentations to my target audience, and gathered their responses through a feedback form included in the appendix. In order to obtain more comprehensive feedback, I played the video for four of my math tutees - Tutees A, E, I, and O - and asked them to provide detailed responses. Tutees A and E, both 12 years old, expressed great interest in the potential applications of Egyptian fractions, with Tutee E even conducting their own research on the topic before the next session. Tutee I, who is 13 years old, enjoyed the video and the mathematics, but did not show a significant interest in exploring the topic further. Tutee O, aged 17, found the video amusing but too easy, and was able to progress to more complicated calculations using Egyptian reasoning and fractions, such as sums of series. They found the topics in the second video more fitting to their level and found the maths quite interesting.

Additionally, I presented the video to School X's maths club, a Year 9 top set, a Year 10 middle set, and a Year 12 A-Level Mathematics class. In order to accommodate for multiple students working at varying rates I distributed a handout containing both the hieroglyphic and hieratic symbols. During the maths club session, I expanded beyond the first video into the second video's content, thanks to the presence of both math enthusiasts and teachers. The feedback from the three teachers present, who seemed highly interested in the topic and asked thought-provoking questions, was very positive, indicating that the video and the maths were enjoyable to them. This led me to believe that these videos could effectively engage individuals beyond my intended audience's age range.

Overall, I collected 40 responses from students, which had overall mean scores of 4.09, 4.02, 4.32, and 4.41 for Interesting, Engaging, Understandable, and Well-structured, respectively. Each category could receive a score from 1 to 5, with 1 representing a low score and 5 representing a high score. The most common responses were 4, 4, 5, and 5 for the same criteria. While 34 respondents positively received the video, two felt neutral about it, three disliked it, and one did not respond. Figure A.3 shows the mean score for each category by age. Based on these results, it can be concluded that the video was generally well-received, with some respondents having stronger positive feelings than others, as evidenced by the mean scores and modal responses. However, it should be noted that a small number of respondents did not like the video, and there were also a few who felt neutral about it, suggesting that the video may not be universally appealing. It is also worth noting that one respondent did not provide any feedback, but this is does not significantly affect the results. Overall, it seems that the video was successful in engaging and communicating with the target audience, but there may be room for improvement to increase its appeal to those who did not respond as positively.

Of the 40 participants, 36 chose to leave further feedback, of which I received 10 entirely positive, 10 entirely negative and 15 mixed comments. Based on these results, it can be concluded that the video elicited diverse opinions among the participants. The fact that a significant proportion of participants provided mixed feedback suggests that the video may have been perceived differently by different individuals, with some finding it more engaging and interesting than others. Several recurring feedback comments indicated that the video was too slow, which led to difficulty in maintaining attention, and the explanation of the example was unclear. Interestingly, other respondents provided feedback that contradicted this, stating that the video was too fast-paced. However, 34 participants reacted positively to the video suggesting that it was generally well-received by the majority of the viewers, with responses describing how the maths was well-explained; the practice questions helped with understanding; the video was intriguing, logical and well-organised in terms of increasing difficulty; and even how the topic was "more interesting than normal maths".

Overall, the results suggest that the videos required some fine-tuning and modification. As a result, I made some changes to my piece of engagement, including adjusting the pace and level of difficulty, improving the clarity of instructions and explanations, and incorporating more interactive elements to keep the viewers engaged. It is important to consider that some learners may benefit from more visual aids, while others may prefer hands-on activities. By taking into account these factors and addressing the feedback received, I was able to improve the videos to better meet the needs of the audience and enhance their learning experience.

From my own observations of these sessions I noted several key points: First, Year 9 and 10 classes were enthusiastic, while Year 12 was more focused on exam work. Second, the topic of Egyptian fractions was noticeably different to typical curriculum topics. Finally, while the video format worked well for a single viewer, allowing them to pause and skip back and forth, additional materials like printed symbol sheets and a PowerPoint presentation for teachers would be necessary if I wanted to expand the engagement activity to a classroom level.

5.2 Obstacles

Throughout my research, I encountered several obstacles that required overcoming to complete the project successfully. One of the significant issues was related to technology and software. I needed to use specific software to create the video and download and store it efficiently, which posed some challenges. The website I used had limited download capacity and occasionally corrupted audio files, resulting in the need for me to re-record certain segments. Another obstacle was related to access to books and resources. I needed to access specific books and papers to ensure the information presented in the video was accurate and credible, but they were not always available to me. Thus, I made several book purchase requests to the library, and eventually received the information that I needed. Additionally, I faced the challenge of limiting information presented in the video, making sure it was not too long or containing heavy content that could overwhelm the audience. I had an overabundance of topics I could potentially discuss but did not want to create uninteresting or tedious videos, so decided to explore some topics in this report. Finally, recording the video professionally posed another challenge. I needed to ensure my tone of voice and volume were appropriate for the audience, making sure I captured their attention and delivered the content effectively. Overcoming these obstacles required a lot of patience, hard work, and perseverance, which were necessary to complete the project successfully.

5.3 Reflections

I am aware that I misjudged the difficulty level between the example and final question in video 1. The example was harder than the final question, and could have been better explained. As a researcher or educator, it is important to recognise the difficulty level of materials and content that can effectively engage and challenge students without overwhelming or discouraging them. The misjudgment of the difficulty level, in this case, highlights the need for careful consideration and assessment of materials during the design process to ensure that the content is appropriately aligned with the knowledge and skills of the target audience. As such, this misjudgment has provided me valuable insights into the development of educational materials and can inform future research and teaching practices to enhance student engagement, learning, and achievement.

If I were to do this project again, I would re-order the video so that the more complex question came after the easier ones, or alternatively choose a different example. I would also give longer or more in-depth explanations of examples. I would also consider adapting my engagement activity to better suit a classroom environment, including attaching the printable handout, and creating a lesson-plan or presentation for teachers to use.

The process of gathering feedback and reflecting on my work has been crucial in improving my ability to create effective and engaging educational materials. By taking into account the feedback received and adapting my approach accordingly, I am confident that I can better engage with my intended audience and promote a deeper understanding of mathematics. This experience has also highlighted the importance of public engagement and the role that educators can play in sharing knowledge and making it accessible to everyone.

Despite the potential for improvements, I was asked by School X to provide my video and handout, so that the topic could be added to their curriculum as part of mathematical exploration! Thus it is clear that my project was successful, and does indeed allow for viewers to engage with the topic of Egyptian fractions, and explore the exciting world of mathematics.

6 Conclusion

In conclusion, the process of creating and testing educational materials can be a challenging and rewarding experience. My research findings demonstrate the value of using video as an engagement medium for teaching and learning mathematical concepts, as it provides a visually engaging and interactive platform for demonstrating the process of working out questions. Through this project, I learned the importance of testing and seeking feedback from a target audience to ensure that the materials are accurately tailored to their needs and abilities. The feedback I received from both individual students and larger groups provided valuable insights into the effectiveness of the materials and the ways in which they could be improved. Additionally, the experience of misjudging the difficulty level of the example versus the final question in video 1 highlighted the importance of reflection and the need to continually evaluate and adjust materials based on feedback and observations. By incorporating these insights into future projects, I hope to create even more effective and engaging educational materials that will inspire and motivate students to pursue their interests in mathematics and other subjects.

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A Appendix

	1 - 5 rating	Comments
How engaging was this video?		
How interesting was the topic?		
Did you understand the maths?		
Was the structure of the video effective?		
	Yes/No	Comments
Do you like the video? Why or why not?		
Could anything be improved? If yes, what?		
Other comments		

Figure A.1: Feedback form

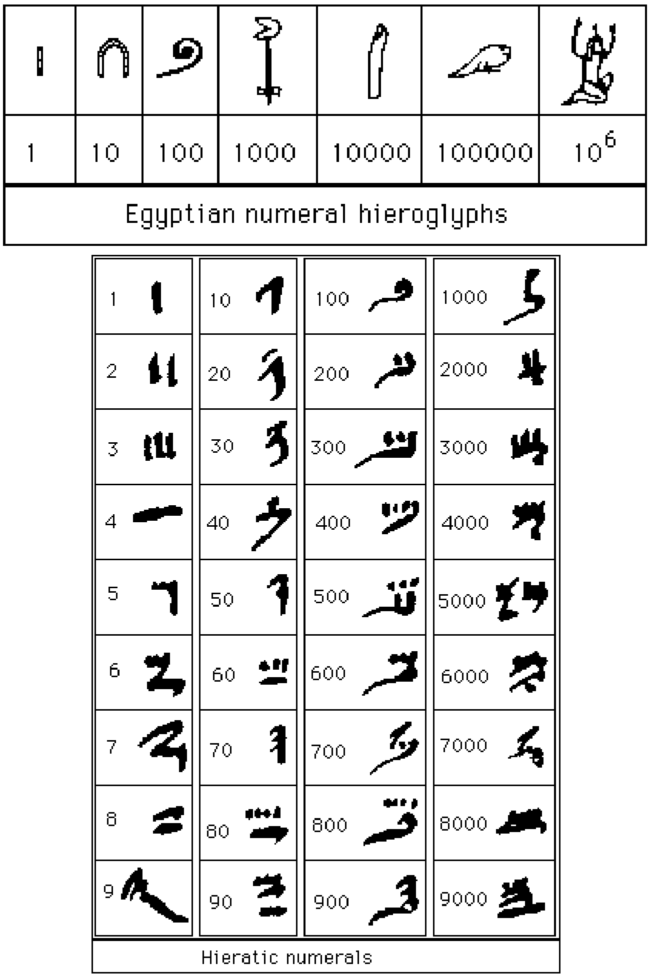

Figure A.2: Handout sheet

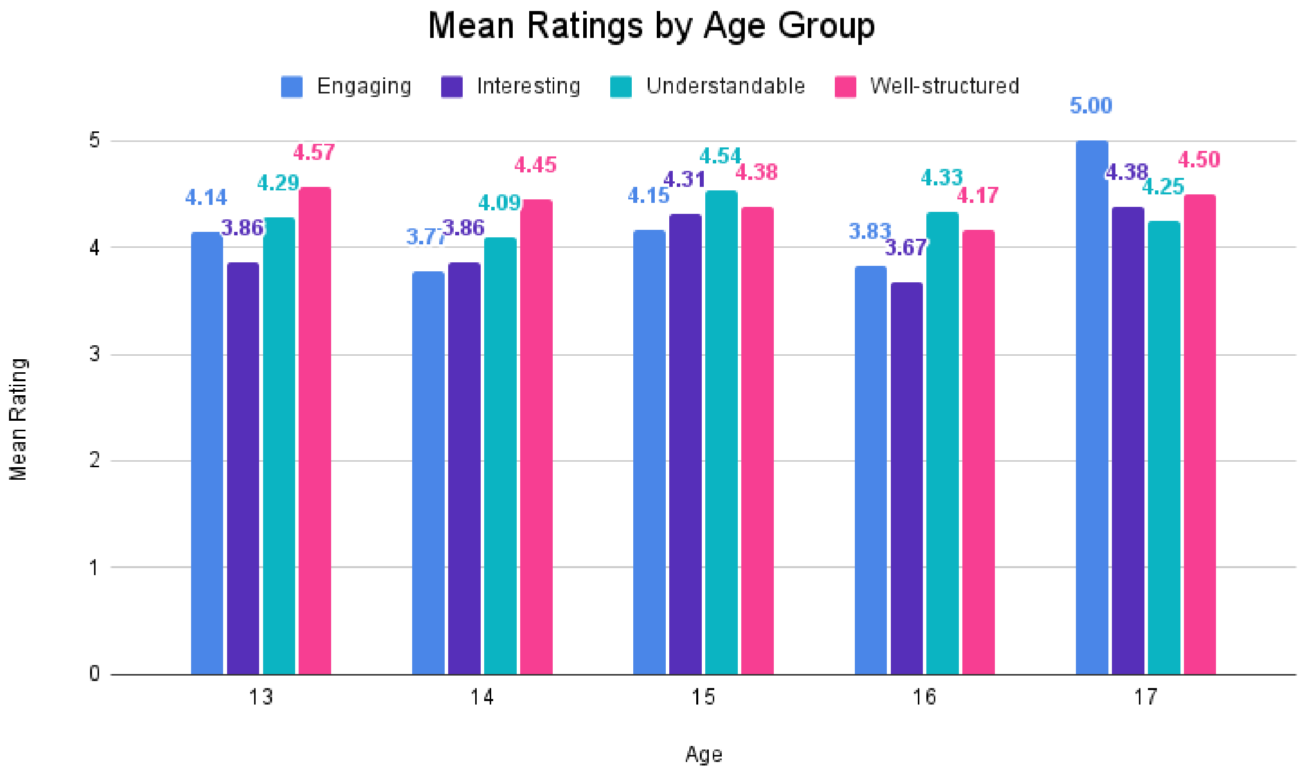

Figure A.3: Column chart of results of questionnaire A.1