UNIVERSITY OF LEEDS

MATH3423 Project in Applied Mathematics

The Dynamics of Accretion Discs

Author:
Milo Kerr (200489745)

Supervisor:
Dr. Evy Kersalé

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1 General Introduction

Following the acceptance of the Copernican Solar System, in which the planets adopt coplanar orbits around the Sun, and the formulation of Kepler's laws, astrophysical discs have been a prominent aspect of astronomy. Since the first proposals of the nebular hypothesis in the 18th century, it has been widely believed that our Solar System is the result of a cooled accretion disc, a disc of condensed gas formed by surrounding material accreting onto a central object. In the case of our Solar System, the central object was the closest star to planet Earth, the Sun. Recent developments in x-ray technology and space telescopy have provided observations of such accretion discs surrounding celestial objects further afield, such as neutron stars and active galactic nuclei (AGN). There is strong evidence that the high energies released from such objects, often x-ray and gamma radiation, can be attributed to the dissipative processes of the orbiting discs. The detailed mechanics of this, however, is still an active area of research.

Massive objects attract surrounding material through gravitational forces. In the absence of angular momentum, such material can accrete directly onto the object, increasing its mass and subsequently its size or density. The potential energy associated with the surrounding mass is consequently released, primarily as electromagnetic radiation. If angular momentum is present in the system however, which is common in the vast majority of accretion flows, centrifugal forces counterbalance gravity causing the formation of an accretion disc. In this case, the mass towards the centre of the disc must redistribute almost the entirety of its angular momentum to the outer regions in order to accrete and release the high energies observed. It is the mechanics behind this redistribution that is of high interest to astrophysicists today.

This project begins by analysing the motion of particles in a gravitational field and discusses how the orbits of such particles are changed in a minimum energy state. We then allow for a transport of angular momentum and mass between particles, based upon the works of Lynden-Bell and Pringle (1974), to highlight some important conditions for further energy dissipation within the system. In search of a mechanism allowing this, we progress in to the subject of fluid dynamics where we expand upon the results of Pringle (1981) and Frank et al. (1992) regarding an accretion disc under shear stress. The discussion of these results influences us to introduce an electrically conducting accretion disc. In line with Balbus and Hawley (1998), we highlight an effective instability within magnetohydrodynamics which could induce the needed transport of angular momentum to result in the observed energy dissipation of accretion discs.

2 Physical Preliminaries

We begin by reminding ourselves of some results from classical mechanics regarding the kinematics, forces and energies associated with particles in space. From our perspective, the size and structure of the particles is irrelevant; we therefore consider them to have mass but no spatial extension. Taylor (2004, p. 13) defines such a particle as a *point particle*. We will later explore the dynamics of accretion discs using the mathematics of a fluid, that is, a continuous medium of particles as opposed to discrete. The results of the following sections, however, provide a useful insight into these dynamics and most are directly applicable.

2.1 Position, Velocity and Acceleration

Consider a particle in three-dimensional space with position vector $\mathbf{r}(t)$ relative to an arbitrary origin in an inertial frame. The *velocity* of the particle, $\mathbf{v}(\mathbf{r}, t)$, is defined as the rate of change of it's position with time, given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}. \tag{2.1}$$

The particles *acceleration*, $\mathbf{a}(\mathbf{r}, t)$, is the rate of change of its velocity such that

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = \ddot{\mathbf{r}}. \tag{2.2}$$

As above, we will commonly use the notation $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ to represent first and second derivatives of a vector $\mathbf{x}$ with respect to time. In the case of cartesian coordinates, the position is given by $\mathbf{r} = (x, y, z)$ and the velocity and acceleration by

$$\mathbf{v} = (v_x, v_y, v_z) = \frac{d}{dt}(x, y, z) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}),$$

$$\mathbf{a} = (a_x, a_y, a_z) = \frac{\partial}{\partial t}(v_x, v_y, v_z) = (\frac{\partial v_x}{\partial t}, \frac{\partial v_y}{\partial t}, \frac{\partial v_z}{\partial t}).$$

The *distance* of the particle from the origin is defined as the length of its position vector, $\|\mathbf{r}\|$, where

$$\|\mathbf{r}\| = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{\mathbf{r}^2}. \tag{2.3}$$

Similarly, the *speed* of the particle is given by $\|\mathbf{v}\|$. Our standard units of length, time and mass are metres (m), seconds (s) and kilograms (kg) respectively. Velocity is therefore measured in $\mathrm{m} \cdot \mathrm{s}^{-1}$ and acceleration in $\mathrm{m} \cdot \mathrm{s}^{-2}$.

2.2 Cylindrical Polar Coordinates

We will commonly adopt cylindrical polar coordinates due to the circular nature of orbital motion. The cylindrical polar coordinate system is an alternative to the three-dimensional Cartesian coordinates. It consists of the values (r, θ, z) where r is the radial distance from the origin, θ is the azimuthal angle measured from an arbitrary reference line, which we take to be the x-axis, and z is the signed height from an arbitrary reference plane as shown in figure 1. The cylindrical coordinates are therefore related to Cartesian coordinates in the following way:

$$x = r\cos\theta; \quad y = r\sin\theta; \quad z = z.$$

We also define the unit vectors

$$\mathbf{e_r} = \cos(\theta)\mathbf{e_x} + \sin(\theta)\mathbf{e_y}, \quad \mathbf{e_{\boldsymbol{\theta}}} = -\sin(\theta)\mathbf{e_x} + \cos(\theta)\mathbf{e_y}, \quad \mathbf{e_z} = \mathbf{e_z},$$

where $\mathbf{e_x}$, $\mathbf{e_y}$ and $\mathbf{e_z}$ are the standard unit vectors given in Cartesian coordinates.

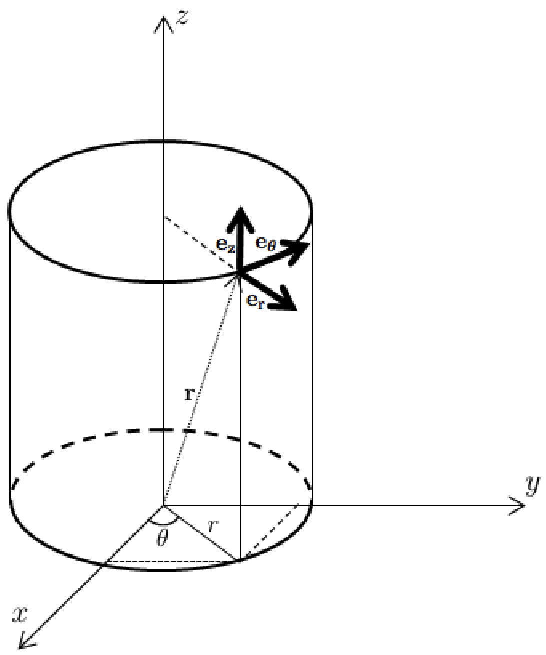

Figure 1: Cylindrical coordinates (r, θ, z) with associated unit vectors $\mathbf{e_r}$, $\mathbf{e_{\boldsymbol{\theta}}}$ and $\mathbf{e_z}$. Adapted from Taylor (2004, p. 136).

The position of a particle with cylindrical coordinates (r, θ, z) is given by

$\mathbf{r} = r\mathbf{e_r} + z\mathbf{e_z}$ and we find the velocity $\mathbf{v} = (v_r, v_\theta, v_z)$ to be

$$\begin{aligned}
\mathbf{v} &= \frac{d}{dt}(r\mathbf{e_r} + z\mathbf{e_z}) \\
&= \frac{d}{dt}\{r[\cos(\theta)\mathbf{e_x} + \sin(\theta)\mathbf{e_y}] + z\mathbf{e_z}\} \\
&= \frac{dr}{dt}[\cos(\theta)\mathbf{e_x} + \sin(\theta)\mathbf{e_y}] \\
&\qquad + r\frac{d\theta}{dt}[-\sin(\theta)\mathbf{e_x} + \cos(\theta)\mathbf{e_y}] + \frac{dz}{dt}\mathbf{e_z},
\end{aligned}$$

therefore,

$$\mathbf{v} = \dot{r}\mathbf{e_r} + r\Omega\mathbf{e_\theta} + \dot{z}\mathbf{e_z} \tag{2.4}$$

where $\Omega = \frac{d\theta}{dt}$ is defined as the *angular velocity* (radians $\cdot\, \mathrm{s}^{-1}$). We highlight that the angular velocity is related to the azimuthal velocity, v_θ, by

$$v_\theta = r\Omega. \tag{2.5}$$

2.3 Newton's Laws of Motion

At the foundations of classical mechanics lies the laws of motion according to Newton, which we will use to establish some key concepts of orbiting objects. Although we are still considering point particles, Newton's laws are also applicable to bodies of mass with spatial components. The laws of motion can be summarised as follows:

1. **Newton's First Law**: The velocity of a particle will remain constant unless the particle is acted on by an external net force $\mathbf{F}$.

2. **Newton's Second Law**: The acceleration of a particle is inversely proportional to its mass, m, and directly proportional to the net force acting upon it, specifically,

   $$\mathbf{F} = m\mathbf{a}. \tag{2.6}$$

   We therefore measure force using the standard unit of newtons (N) where $\mathrm{N} = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$.

3. **Newton's Third Law**: For every force, $\mathbf{F_1}$, that a particle exerts on a second particle, the second particle simultaneously exerts a force, $\mathbf{F_2}$, on the first such that $\mathbf{F_2} = -\mathbf{F_1}$, that is, an equal and opposite force.

These laws are extensively used in the derivation of the governing formulae for the motion of particles and fluids.

2.4 Linear and Angular Momentum

We define the *linear momentum* of a particle with mass m as the vector $\mathbf{L}$ such that

$$\mathbf{L} = m\mathbf{v},$$

while the *angular momentum*, $\mathbf{H}$, is given by

$$\mathbf{H} = \mathbf{r} \times \mathbf{L},$$

where both quantities are measured in newton metre seconds ($\mathrm{N} \cdot \mathrm{m} \cdot \mathrm{s}$). If m is constant, equation (2.6) can therefore be written in the form

$$\mathbf{F} = \frac{\partial \mathbf{L}}{\partial t}. \tag{2.7}$$

As a direct consequence of Newton's laws of motion, they are both *conserved quantities*, meaning that there totals within a closed system can not change unless acted on by external forces.

2.5 Newton's Law of Gravity and Gravitational Fields

Newton's law of gravity states that the *gravitational force*, $\mathbf{F_g}(\mathbf{r})$, that a particle with mass m_1 exerts on a second particle with mass m_2 is given by

$$\mathbf{F_g} = -\frac{G m_1 m_2}{\|\mathbf{r}\|^2}\hat{\mathbf{r}} \tag{2.8}$$

where G is the *universal gravitational constant* (approximately $6.67\times10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$), $\mathbf{r}$ is the position vector of m_2 relative to m_1 and $\hat{\mathbf{r}}$ is a unit vector pointing in the direction from m_1 to m_2 as shown in figure 2.

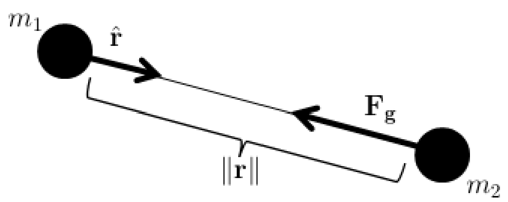

Figure 2: The gravitational force, $\mathbf{F_g}$, of m_1 on m_2.

This notion can be extended by the introduction of a *gravitational field*, $\mathbf{g}(\mathbf{r})$, a vector field that describes the gravitational force exerted by a mass m on a particle in space per unit mass. From here on, we use the word '*specific*' in place of '*per unit mass*'. This vector field is given by

$$\mathbf{g} = -\frac{G m}{\|\mathbf{r}\|^2}\hat{\mathbf{r}}, \tag{2.9}$$

which is also equal to the gravitational acceleration of a particle at the point $\mathbf{r}$ due to the mass m by equation (2.6).

2.6 Equations of Energy

In the forthcoming sections, we will analyse the energy associated with orbiting particles. The two types we will focus on are kinetic energy arising from motion and potential energy arising from gravity. The *kinetic energy*, E_k, of a particle with mass m and velocity $\mathbf{v}$ is given by

$$E_k = \frac{m\mathbf{v}^2}{2}. \tag{2.10}$$

measured in joules (J) where J $=$ N $\cdot$ m. Given a gravitational field $\mathbf{g}(\mathbf{r})$, there exists a *gravitational potential field*, $\varphi(\mathbf{r})$, such that

$$\mathbf{g} = -\nabla\varphi. \tag{2.11}$$

At each point in the field, φ is equal to the *specific gravitational potential energy* (J $\cdot$ kg^{-1}) a particle at that point would have.

3 The Motion of Particles in Space

Given the preliminary laws of section 2, we can introduce the concept of motion in space by considering the position of one mass with respect to another due to gravitational forces. We then continue by analysing how this position changes when the mass assumes a least energy state.

3.1 Two Body Problem

Let us introduce two masses, m_1 and m_2, with position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ relative to an origin O in a fixed inertial frame as shown in figure 3. We follow the works of Murray and Dermott (1999, pp. 22-24) to analyse the position of the mass m_2 with respect to m_1, denoted by the vector $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$. A unit vector in the direction of $\mathbf{r}$ is thus given by $\hat{\mathbf{r}} = \mathbf{r}/\|\mathbf{r}\|$.

Masses m_1 and m_2 are attracted to each other by gravitational forces $\mathbf{F_{g_1}}$ and $\mathbf{F_{g_2}}$; they therefore have gravitational accelerations $\ddot{\mathbf{r}}_\mathbf{1}$ and $\ddot{\mathbf{r}}_\mathbf{2}$ respectively, in accordance with equation (2.9), given by

$$\ddot{\mathbf{r}}_\mathbf{1} = \frac{Gm_2}{\|\mathbf{r}\|^2}\hat{\mathbf{r}} = \frac{Gm_2}{\|\mathbf{r}\|^3}\mathbf{r},$$

$$\ddot{\mathbf{r}}_\mathbf{2} = -\frac{Gm_1}{\|\mathbf{r}\|^2}\hat{\mathbf{r}} = -\frac{Gm_1}{\|\mathbf{r}\|^3}\mathbf{r}.$$

Relative to m_1, m_2 has acceleration

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_\mathbf{2} - \ddot{\mathbf{r}}_\mathbf{1} = -\frac{Gm_1}{\|\mathbf{r}\|^3}\mathbf{r} - \frac{Gm_2}{\|\mathbf{r}\|^3}\mathbf{r}$$

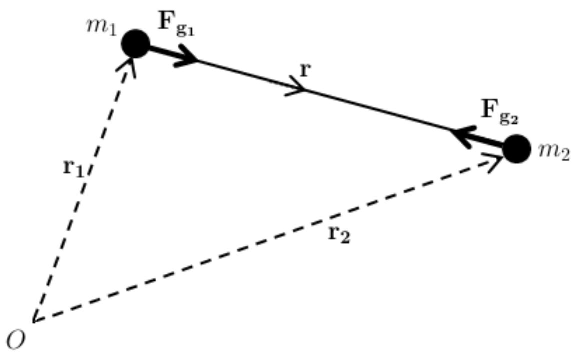

Figure 3: The gravitational forces associated with two masses, m_1 and m_2, in space. Adapted from Murray and Dermott (1999, p. 22).

which can be rearranged to obtain the second order differential equation

$$\ddot{\mathbf{r}} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = \mathbf{0}, \tag{3.1}$$

where $\mu = G(m_1 + m_2)$ is the *standard gravitational parameter*. Equation (3.1) is known as the *equation of relative motion*; it can be solved to obtain the position of m_2 with respect to m_1 (e.g. see Taylor (2004, chapter 8) or Knudsen and Hjorth (2000, chapter 14)). The solution of this differential equation, however, will currently be of little use to us and we will obtain similar properties in due course when considering minimum energy states. More importantly, it does give us an extraordinary consequence of motion due to gravity. Taking the cross product of $\mathbf{r}$ with equation (3.1), we find

$$\mathbf{r} \times \ddot{\mathbf{r}} + \frac{\mu}{\|\mathbf{r}\|^3}(\mathbf{r} \times \mathbf{r}) = \mathbf{0}$$

$$\Rightarrow \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} \tag{3.2}$$

and, since

$$\frac{\partial}{\partial t}(\mathbf{r} \times \dot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) + (\mathbf{r} \times \ddot{\mathbf{r}}),$$

integrating (3.2) with respect to time gives

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} \tag{3.3}$$

for some constant vector $\mathbf{h}$ perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, defined as the *specific relative angular momentum vector* ($\mathrm{m}^2 \cdot \mathrm{s}^{-1}$). This implies that the position and velocity vectors of an orbiting particle always lie in the same plane, defined as the *orbit plane*. Such a particle is said to be in *Keplerian orbit* after the German mathematician Johannes Kepler (1571-1630) who initially published the laws of planetary motion highlighting this coplanar property. The set up of our problem can therefore be revised.

3.2 Reduction to a One Body Problem

We continue by defining the origin of a cylindrical coordinate system as the position of the particle m_2 and restrict the orbit plane to $z = 0$. Our three-dimensional, two body problem has therefore been reduced to a two-dimensional, one body problem for m_1 in the presence of a central, axisymmetrical gravitational field due to the mass m_2. The position of m_1 satisfies equation (3.1) and is given by $\mathbf{r} = r\mathbf{e_r}$. We will commonly take the central mass to be much larger than the orbiting mass; to highlight this, we rename the central mass as M and the orbiting mass as m. In this case, the specific relative angular momentum is given by

$$\begin{aligned}
\mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}} \\
&= \mathbf{r} \times (\dot{r}\mathbf{e_r} + r\Omega\mathbf{e}_{\boldsymbol{\theta}}) \\
&= r\mathbf{e_r} \times r\Omega\mathbf{e}_{\boldsymbol{\theta}} \\
&= r^2\Omega\sin(\phi)\mathbf{e_z}
\end{aligned}$$

where ϕ is the angle between $\mathbf{e_r}$ and $\mathbf{e}_{\boldsymbol{\theta}}$, however, these two unit vectors are perpendicular so

$$\mathbf{h} = r^2\Omega\mathbf{e_z}.$$

The system of an orbiting point particle in a gravitational field therefore has specific relative angular momentum perpendicular to its plane of orbit with magnitude $h = r^2\Omega$.

3.3 Minimum Energy State

By the principle of minimum energy, the internal energy of a closed system will decrease and approach a minimum value at equilibrium. We apply this principle to our system by minimising the total energy and analyse how this affects the orbiting particles motion. The gravitational potential field of a central mass M satisfies

$$\begin{aligned}
\nabla\varphi(\mathbf{r}) &= \frac{GM}{\|\mathbf{r}\|^3}\mathbf{r} \\
&= \frac{GM}{(r^2 + z^2)^{3/2}}\mathbf{r}.
\end{aligned}$$

We see

$$\begin{aligned}
\nabla\left[-\frac{GM}{(r^2 + z^2)^{1/2}}\right] &= \frac{2rGM}{2(r^2 + z^2)^{3/2}}\hat{\mathbf{r}} + \frac{2zGM}{2(r^2 + z^2)^{3/2}}\hat{\mathbf{z}} \\
&= \frac{GM}{(r^2 + z^2)^{3/2}}(r\hat{\mathbf{r}} + z\hat{\mathbf{z}}) \\
&= \frac{GM}{(r^2 + z^2)^{3/2}}\mathbf{r}
\end{aligned}$$

thus we conclude that the gravitational potential field is given in cylindrical coordinates by

$$\varphi(r, z) = -\frac{GM}{(r^2 + z^2)^{1/2}}. \tag{3.4}$$

The *total specific energy*, ϵ, of the mass m as described in section 3.2 is therefore given by

$$\begin{aligned}
\epsilon &= \frac{E_k}{m} + \varphi(r, z) \\
&= \frac{\mathbf{v}^2}{2} - \frac{GM}{(r^2 + z^2)^{1/2}} \\
&= \frac{1}{2}\left(\dot{r}^2 + r^2\Omega^2 + \dot{z}^2\right) - \frac{GM}{(r^2 + z^2)^{1/2}} \\
&= \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \frac{h^2}{2r^2} - \frac{GM}{(r^2 + z^2)^{1/2}}.
\end{aligned}$$

Our coordinates, however, were defined such that the particle orbits on the plane $z = 0$, and consequently $\dot{z} = 0$, thus

$$\epsilon = \frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} - \frac{GM}{r}.$$

The minimum energy state of the system is therefore achieved when $\dot{r} = 0$ and the value of $\frac{h^2}{2r^2} - \frac{GM}{r}$ is minimised. We note that h is constant as a consequence of equation (3.3) leaving only r to minimise, giving

$$\epsilon_{\text{min}} = \frac{h^2}{2r_{\text{min}}^2} - \frac{GM}{r_{\text{min}}} \tag{3.5}$$

where r_{min} satisfies

$$\begin{aligned}
\left.\frac{\partial}{\partial r}\right|_{r=r_{\text{min}}} \left(\frac{h^2}{2r^2} - \frac{GM}{r}\right) &= 0 \\
\Rightarrow -\frac{h^2}{r_{\text{min}}^3} + \frac{GM}{r_{\text{min}}^2} &= 0 \\
\Rightarrow r_{\text{min}} &= \frac{h^2}{GM}. \qquad (3.6)
\end{aligned}$$

Therefore, an orbiting particle in equilibrium around a central point particle has a circular, Keplerian orbit in the plane $z = 0$ with radius $r = r_{\text{min}} = \frac{h^2}{GM}$. We note that in this case,

$$h = h(r) = (GMr)^{\frac{1}{2}}, \tag{3.7}$$

$$\epsilon_{\text{min}} = \epsilon_{\text{min}}(h) = -\frac{1}{2}\left(\frac{GM}{h}\right)^2, \tag{3.8}$$

$$\Omega = \Omega(r) = \left(\frac{GM}{r^3}\right)^{\frac{1}{2}}. \tag{3.9}$$

4 Further Dissipation of Energy

We now turn our attention to how further energy can be dissipated from a system of orbiting particles in equilibrium within the gravitational field of a central mass. A simple scale analysis given by Frank et al. (1992, p. 1) shows that the gravitational potential energy released during accretion has the capability of being twenty times that of nuclear fusion. In this section, we follow a similar approach to Lynden-Bell and Pringle (1974) in order to gain an insight into how such an efficient dissipation can be achieved by considering the transportation of angular momentum and mass. This will highlight some important conditions that must be met in order for high levels of energy to be released that we can later apply to a continuous accretion disc.

4.1 Angular Momentum Transportation

Let us introduce two particles, m_1 and m_2, in the presence of a fixed gravitational field due to a central mass M. We assume that $M \gg m_1, m_2$ so the orbits of the particles are only influenced by the gravitational field of the central mass. In equilibrium, the particles m_1 and m_2 have circular orbits with radii r_1 and r_2 respectively, in accordance with equation (3.6). Consequently, they also have angular velocities $\Omega_1(r_1)$ and $\Omega_2(r_2)$ and specific relative angular momenta $h_1(r_1)$ and $h_2(r_2)$. We assume that $r_1 < r_2$ as shown below.

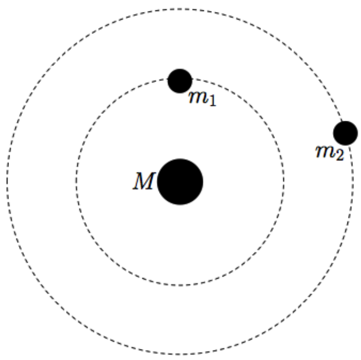

Figure 4: Two masses, m_1 and m_2, in the fixed gravitational potential field of a central mass M.

To establish conditions on how the system can achieve an efficient dissipation of energy, we analyse how the total energy of the two particles can be reduced by allowing them to exchange angular momentum. In accordance with conservation laws discussed in section 2.4 however, we keep the total angular momentum, H, to be constant. The total energy, E, of the two particles is given by

$$E = m_1 \epsilon_{\min}(h_1) + m_2 \epsilon_{\min}(h_2) \tag{4.1}$$

where $\epsilon_{\min}(h)$ is defined as the energy of the particle in equilibrium given by equation (3.8). The total angular momentum of the two particles, defining H_1 and H_2 as the angular momenta for m_1 and m_2 respectively, is

$$H = H_1 + H_2 = m_1 h_1 + m_2 h_2.$$

We now consider a small change in the specific angular momenta h_1 and h_2. This gives a change in the total energy equal to

$$dE = m_1 dh_1 \left.\frac{d\epsilon_{\min}}{dh}\right|_{h=h_1} + m_2 dh_2 \left.\frac{d\epsilon_{\min}}{dh}\right|_{h=h_2}.$$

From equation (3.5) we find

$$\frac{d\epsilon_{\min}}{dh} = \frac{h}{r^2} = \Omega,$$

therefore,

$$dE = m_1 dh_1 \Omega_1 + m_2 dh_2 \Omega_2 = dH_1 \Omega_1 + dH_2 \Omega_2$$

and imposing the constraint that H is constant, which consequently implies that $dH_1 + dH_2 = 0$, we see

$$dE = dH_1(\Omega_1 - \Omega_2).$$

As we defined $r_1 < r_2$ it must follow that $\Omega_1 > \Omega_2$. It is therefore the case that $dE < 0$ if and only if $dH_1 < 0$ and consequently $dH_2 > 0$. We conclude that the system of two particles can dissipate energy if and only if angular momentum is transported outwards from m_1 to m_2.

4.2 Mass Transportation

As a continuation, we now consider the possibility of further energy dissipation when allowing an exchange of mass between the two particles, keeping the total mass, $M = m_1 + m_2$, to be constant. In this case, our constraints are

$$dM = dm_1 + dm_2 = 0,$$

$$dH = dH_1 + dH_2 = 0$$

where dH_1 and dH_2 are now given by

$$dH_i = m_i dh_i + h_i dm_i$$

for $i = 1, 2$. From the expression for the total energy of the two particles given in equation (4.1) we see

$$\begin{aligned} dE &= dm_1 \epsilon_{\min}(h_1) + m_1 dh_1 \Omega_1 + dm_2 \epsilon_{\min}(h_2) + m_2 dh_2 \Omega_2 \\ &= dm_1[\epsilon_{\min}(h_1) - h_1\Omega_1] + dm_2[\epsilon_{\min}(h_2) - h_2\Omega_2] + dH_1\Omega_1 + dH_2\Omega_2 \end{aligned}$$

and imposing our constraints we find

$$dE = dm_1\{[\epsilon_{\min}(h_1) - h_1\Omega_1] - [\epsilon_{\min}(h_2) - h_2\Omega_2]\} + dH_1(\Omega_1 - \Omega_2).$$

The second term in this expression is in agreement with our analysis in section 4.1. For the first term, we see

$$\begin{aligned} \frac{d}{dr}[\epsilon_{\min}(h) - h\Omega] &= \frac{dh}{dr}\frac{d\epsilon_{\min}}{dh} - \frac{dh}{dr}\Omega - h\frac{d\Omega}{dr} \\ &= \frac{GM}{2r^2} - \frac{GM}{2r^2} - h\frac{d\Omega}{dr} \\ &= -h\frac{d\Omega}{dr} > 0 \end{aligned}$$

since $\frac{d\Omega}{dr} < 0$, hence $[\epsilon_{\min}(h_1) - h_1\Omega_1] - [\epsilon_{\min}(h_2) - h_2\Omega_2] < 0$ and it follows that $dm_1\{[\epsilon_{\min}(h_1) - h_1\Omega_1] - [\epsilon_{\min}(h_2) - h_2\Omega_2]\} < 0$ if and only if $dm_1 > 0$ and subsequently $dm_2 < 0$. Energy is therefore dissipated further if mass is transported inwards to smaller radii.

4.3 Summary of Discrete Particle Analysis

In the previous sections, we have established some incredible results regarding motion around a central, massive point particle. Gravitational forces influence surrounding particles to take coplanar orbits and a minimum energy configuration has shown further that the orbits are circular in a state of equilibrium. To further extract energy from an equilibrium state, there must be a process that redistributes angular momentum radially away from the central mass, while the mass of the particles must be transported inwards. In this case, a minimum energy configuration would see the entire mass of the system accumulated at the centre, while a particle with infinitesimal mass at infinity carries the entire angular momentum.

Of course, this result is based on the existence of a mechanism that will allow angular momentum and mass to be freely transported, while energy can dissipate at will, which is not the case unless we introduce a mechanism that will allow this. As a physical example, consider the orbit of the Moon around the Earth. Loosely speaking, angular momentum can only be exchanged to the Moon from the Earth as the tides of the sea create a *torque* allowing it to do so. Without this torque, angular momentum would not be exchanged as it currently is. To conclude, dissipation of energy can only exist in an accretion disc if there is in fact a process transporting mass inwards and angular momentum outwards. It is the formulation of this process that has taken centre stage in accretion disc research over the past forty years. In order to discuss the developments of this research, we must turn to the subject of fluid dynamics which will allow us to model the mathematics of accretion discs using continuum mechanics.

5 Astrophysical Fluid Dynamics Equations

The subject of fluid dynamics concerns the flow of liquids and gases in space, which can be modelled mathematically using a small number of fundamental equations and assumptions, which we will derive in the following section. In order to make the transition from the dynamics of particles to that of a fluid however, we must introduce the notion of a fluid element.

5.1 Introduction to a Fluid Element

Due to the large-scale molecular structure of a fluid, it is clearly impractical, and most likely impossible, to accurately calculate the motion of each particle, especially on the scale of astrophysical fluids. As an alternative to this discrete, molecular classification, we consider a fluid to be a continuous structure of small volumes, δV, known as *fluid elements* in which physical properties, such as velocity and pressure, are considered well-defined. These volumes can be considered as points.

The motion of a fluid is defined by the *velocity field* $\mathbf{v}(\mathbf{r}, t)$. Here, $\mathbf{v}$ can be thought of as the average velocity of all molecules in the fluid element δV centered at a fixed point P with position vector $\mathbf{r}$. We define the *density*, ρ, of the fluid element as

$$\rho(\mathbf{r}, t) = \frac{\text{mass in } \delta V}{\delta V} \tag{5.1}$$

with standard units $\text{kg} \cdot \text{m}^{-3}$. The mass, therefore, of a fluid contained in a volume V is given by the volume integral

$$M = \int_V \rho \, dV. \tag{5.2}$$

5.2 Lagrangian Description of Fluids

In introducing the concept of a fluid element, we have altered the definition of the vector $\mathbf{r}$, and subsequently $\mathbf{v}$, from the position of a moving particle at time t to a fixed point within a fluid, and $\mathbf{v}$ as the velocity of the fluid at this point. The rate of change of a scalar f or vector $\mathbf{F}$ can also be calculated at this fixed point using the definitions of time derivatives we are accustomed to,

$$\frac{\partial f}{\partial t} \text{ and } \frac{\partial \mathbf{F}}{\partial t},$$

known as the *Eulerian time derivatives*. To calculate the rate of change of a scalar or vector at a point moving with the fluid, however, the *Lagrangian time derivatives*, $\frac{D}{Dt}$, are used, where

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla)f \text{ and } \frac{D\mathbf{F}}{Dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{F}. \tag{5.3}$$

Appendix A gives expressions for this operator in cylindrical coordinates. As a consequence of this definition, the acceleration of a fluid is given by the Lagrangian derivative

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{5.4}$$

These expressions can be derived from first principles using the chain rule (e.g. see Acheson (1990, p. 4)).

5.3 Conservation of Mass

One important property of a fluid, and indeed any closed system, is that the total mass it holds is conserved. Following the formalisation of this concept by Paterson (1983, chapter 4), we consider a volume of fluid V, fixed in space, enclosed by a permeable surface S with outward pointing normal $\mathbf{n}$. Let M be the total mass of the volume of fluid in accordance with equation (5.2). The only way M can change is if mass is transported in or out of the volume V, that is, if mass passes through the surface S. The rate at which this happens is given by the *mass flux*

$$-\int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS,$$

measured in $\mathrm{kg} \cdot \mathrm{s}^{-1}$, therefore,

$$\frac{dM}{dt} = -\int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS. \tag{5.5}$$

We emphasise here the use of the Eulerian time derivate since V is fixed. The negative sign is due to the outward pointing normal. In an attempt to clarify this, consider a situation where fluid flows only out of the volume in the direction of the normal $\mathbf{n}$ giving $\mathbf{v} \cdot \mathbf{n} > 0$. Then equation (5.5) implies that $\frac{dM}{dt} < 0$, i.e, the mass decreases, which seems intuitive. By the definition of M, equation (5.5) becomes

$$\frac{d}{dt} \int_V \rho \, dV = - \int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS \;\Rightarrow \int_V \frac{\partial \rho}{\partial t} \, dV = - \int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS$$

since the volume V is fixed. Applying the divergence theorem given in appendix B we find

$$- \int_V \nabla \cdot (\rho \mathbf{v}) \, dV = \int_V \frac{\partial \rho}{\partial t} \, dV \Leftrightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] = 0.$$

As V is chosen arbitrarily, this must hold for all volumes V, therefore

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{5.6}$$

which we define as the *equation of mass conservation*. Alternatively, using $\nabla \cdot (\rho \mathbf{v}) = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$, it can be expressed in the Lagrangian form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{5.7}$$

This is our first fundamental equation of fluid dynamics. In order to derive the second, we must discuss the forces which can act upon a fluid.

5.4 Forces on a Fluid

There are two types of forces that can act upon a volume of fluid. The first are forces which act upon the entire volume, such as gravity, known as *body forces*. We shall denote these forces per unit mass as $\mathbf{F}(\mathbf{x}, t)$, so the total body forces acting upon a volume of fluid V is

$$\int_V \mathbf{F} \rho \, dV.$$

The second type are forces acting upon the surface of the volume of fluid, known as *surface forces*. These can be described in the form of a *stress tensor* $\boldsymbol{\sigma}$, where σ_{ij} is defined by Batchelor (1967, p. 10) as

$$\text{“}\textit{the } i\textit{th-component of the force per unit area exerted}$$
$$\textit{across a plane surface element normal to the } j\textit{-direction}''. \tag{5.8}$$

We will informally derive an expression for the stress tensor in the Navier-Stokes context, however, an in-depth proof can be obtained (e.g. see Batchelor (1967, pp. 137-147)).

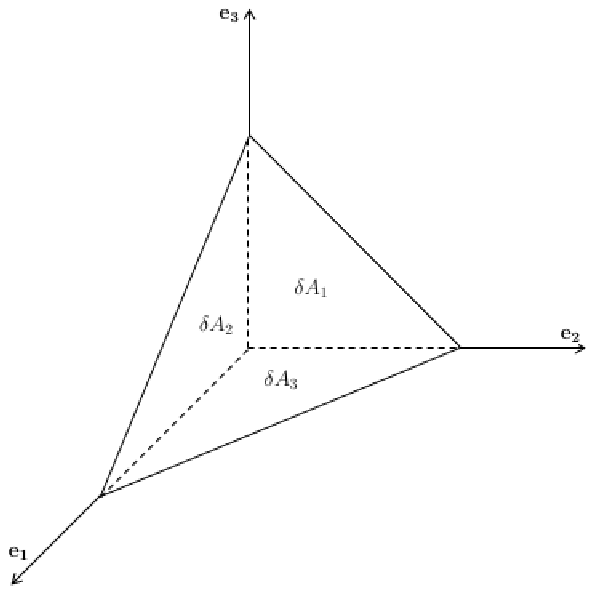

Figure 5: Tetrahedral volume of fluid in orthogonal coordinate system. Adapted from Paterson (1983, p. 97).

Consider the tetrahedral fluid element, δV, shown in figure 5 in an orthogonal coordinate system $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Let δA be the area of the large face opposite the origin with outward normal $\mathbf{n} = (n_1, n_2, n_3)$, and the other faces have areas δA_1, δA_2 and δA_3 as labelled, each with outward normals $-\mathbf{e_1}$, $-\mathbf{e_2}$ and $-\mathbf{e_3}$ respectively. We define the instantaneous ith-component of the force per unit area on the surface δA as τ_i, so the surface force on δA is $\delta A \tau_i$. The ith-component of the surface force on δA_1 is $-\sigma_{i1}\delta A_1$ by the definition given in (5.8), and similarly $-\sigma_{i2}\delta A_2$ and $-\sigma_{i3}\delta A_3$ on δA_2 and δA_3 respectively. By the orthogonality of the coordinate system,

$$\delta A_i = \mathbf{e_i} \cdot \mathbf{n} \delta A = n_i \delta A$$

so the sum of the ith-components of surface force on the volume δV can be written

$$\begin{aligned} \tau_i \delta A - (\delta A_1 \sigma_{i1} + \delta A_2 \sigma_{i2} + \delta A_2 \sigma_{i2}) &= [\tau_i - (\sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3)] \delta A \\ &= (\tau_i - \sum_{j=1}^{3} \sigma_{ij} n_i) \delta A. \end{aligned}$$

At any fixed point in time, by Newton's Second Law, this surface force plus any body force must be equal to the acceleration times the mass of the fluid element such that

$$ma_i = (\tau_i - \sum_{j=1}^{3} \sigma_{ij} n_i)\delta A + F_i.$$

Letting l be a typical side length of the fluid element, we know $m \sim \delta V \sim l^3$ and $\delta A \sim l^2$, so as $l \to 0$, the surface force term dominates while the other terms tend towards zero. Therefore, the ith-component of the surface force on a volume of fluid with outward normal vector $\mathbf{n}$ is given by

$$\tau_i = \sum_{j=1}^{3} \sigma_{ij} n_i = \boldsymbol{\sigma_i} \cdot \mathbf{n} \tag{5.9}$$

where $\boldsymbol{\sigma_i} = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$. A similar argument can show that the stress tensor is symmetric, that is, $\sigma_{ij} = \sigma_{ji}$ for all i, j. The tensor can therefore be represented in the form of the 3x3 matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}.$$

The components along the diagonal are *normal stresses* and those off-diagonal *shear stresses* since they only arise due to a shearing motion between adjacent fluid elements moving relative to each other. In a fluid at rest, therefore, the only surface forces present are normal forces contracting the fluid element. The amount of force is dependent on the steady conditions outside of the fluid element, which we can assume to be identical in each direction since we have taken the fluid element to be analogous to a point in space. Therefore at rest, the stress tensor becomes

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

where p is defined as the *pressure* on the fluid element measured in pascals ($\mathrm{Pa} = \mathrm{N} \cdot \mathrm{m}^{-2}$). The derivation of the stress tensor for a fluid in motion is dependent upon pressure remaining as the negative mean of normal stresses, while introducing a molecular shear stress due to motion. To highlight the source of this stress, consider a surface within a fluid where the horizontal velocities above the surface are faster than those below as depicted in figure 6.

Although we consider a fluid to be a continuous medium with a bulk velocity, the motion of molecules within the medium is close to random. Therefore, molecules with horizontal velocities U are free to permeate through the surface S into the region of higher horizontal velocities, causing the average

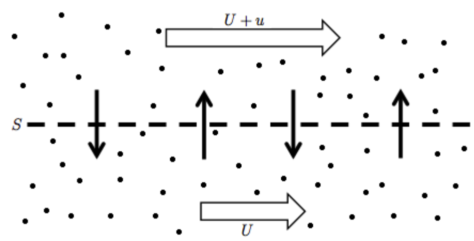

Figure 6: The principle of molecular viscosity. Adapted from Paterson (1983, p. 129).

velocity in this region to reduce slightly. Similarly, those from the region of higher velocites can move through S and increase the average velocity of the slower region. For single molecules this effect is extremely negligible, but on the scale of a fluid consisting of millions of freely moving molecules, its effects must be taken into account by treating them as a surface force. This is the basis of shear stress due to molecular viscosity.

Introducing viscosity into our expression for the stress tensor $\boldsymbol{\sigma}$, we will assume that in cartesian coordinates it takes the form

$$\sigma_{ij} = \begin{cases} \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) & \text{if } i \neq j; \\ p + 2\mu\left[\frac{\partial u_i}{\partial x_i} - \frac{1}{3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\right] & \text{if } i = j, \end{cases} \tag{5.10}$$

where μ is defined as the *dynamic viscosity* ($\mathrm{Pa} \cdot \mathrm{s}^{-1}$) which takes different values dependent on physical properties of the fluid itself. For example, Paterson (1983) gives the dynamic viscosity of air at 288 kelvin to be 1.8×10^-5 while for olive oil it is given as 0.10. The dynamic viscosity is related to the *kinematic viscosity*, ν, with units $\mathrm{m}^2 \cdot \mathrm{s}^{-1}$, by

$$\mu = \rho\nu. \tag{5.11}$$

Frank et al. (1992, p. 59) states that $\nu \sim \lambda v_{\mathrm{mol}}$ where λ is the mean distance that a free molecule travels before colliding in the fluid and v_{mol} is the mean speed of a free molecule in the fluid.

Our expression for the stress tensor in cartesian coordinates can be extended to other coordinate systems by defining the tensor in the form

$$\boldsymbol{\sigma} = p\mathbf{I} + \mathbf{T}$$

where $\mathbf{I}$ is the 3x3 identity matrix and $\mathbf{T}$ is defined as the *deviatoric stress tensor* given by

$$\mathbf{T} = 2\mu\left[\frac{1}{2}(\nabla\mathbf{v}) + \frac{1}{2}(\nabla\mathbf{v})^T - \frac{1}{3}(\nabla\cdot\mathbf{v})\mathbf{I}\right]. \tag{5.12}$$

where $\nabla\mathbf{v}$ is defined as the *tensor gradient.* An expression for this operator in cylindrical coordinates is given in appendix A.

5.5 Equation of Motion

We now turn our attention to the momentum of a volume of fluid given by

$$\int_V \mathbf{v}\rho\, dV.$$

Paterson (1983, p. 127) describes three ways in which the momentum of a volume of fluid can be changed in accordance with Newton's second law:

1. By an inflow of momentum through the surface S;
2. By body forces acting on the volume V;
3. By surface forces acting on S.

Imposing this, we therefore find that the rate of change of the ith-component of momentum must satisfy

$$\frac{d}{dt}\int_V \rho v_i\, dV = -\int_S \rho v_i \mathbf{v}\cdot\mathbf{n}\, dS + \int_V F_i\rho\, dV + \int_S \boldsymbol{\sigma_i}\cdot\mathbf{n}\, dS$$

and applying the divergence theorem to the surface integrals gives

$$\int_V \frac{\partial}{\partial t}(\rho v_i)\, dV = -\int_V \nabla\cdot(\rho v_i\mathbf{v})\, dV + \int_V F_i\rho\, dV + \int_V \nabla\cdot\boldsymbol{\sigma_i}\, dV$$

$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(\rho v_i) + \nabla\cdot(\rho v_i\mathbf{v}) - F_i\rho - \nabla\cdot\boldsymbol{\sigma_i}\right] dV.$$

Since V is arbitrary, and expanding the stress component into pressure and deviatoric parts, we must have

$$0 = \frac{\partial}{\partial t}(\rho v_i) + \nabla\cdot(\rho v_i\mathbf{v}) - F_i\rho + \frac{\partial p}{\partial x_i} - \nabla\cdot\mathbf{T_i}. \tag{5.13}$$

Now, expanding the first two terms using the product rule, we see

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) + \nabla(\rho v_i\mathbf{v}) &= \left[\rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial\rho}{\partial t}\right] + \left[v_i\nabla\cdot(\rho\mathbf{v}) + \rho(\mathbf{v}\cdot\nabla)v_i\right]\\ &= \rho\left[\frac{\partial v_i}{\partial t} + (\mathbf{v}\cdot\nabla)v_i\right] + v_i\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right]\\ &= \rho\frac{Dv_i}{Dt}\end{aligned}$$

by equating the second term to zero using the equation of mass conservation equation. Substituting this into equation (5.13) we therefore find

$$\rho \frac{Dv_i}{Dt} = F_i \rho - \frac{\partial p}{\partial x_i} + \nabla \cdot \mathbf{T_i}$$

or

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}\rho - \nabla p + \nabla \cdot \mathbf{T} \tag{5.14}$$

which is the *Navier-Stokes equation of motion*. If we take $\mathbf{T}$ to be defined as a shear stress by equation (5.12), this equation can be written

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F}\rho - \nabla p + \mu \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right]. \tag{5.15}$$

It is important to note that although we have taken shear stress as a result of molecular viscosity, its terms within the equation of motion may be used to parameterise other origins of surface stress. The Navier-Stokes equation is our second fundamental equation of fluid dynamics. We are now in a position where we can analyse the mechanics of accretion discs.

6 Accretion Discs with Shear Stress

Let us consider a central mass at the origin $r = z = 0$ in a cylindrical coordinate system surrounded by a continuous disc of matter which we treat as a fluid. Our analysis in section 3 leads us to suspect that gravity will force the matter within the disc into a coplanar orbit. In the case of a continuous fluid where pressure is present, it can not be assumed that a perfectly flat disc will be formed. We will, however, make the reasonable assumption that the disc is thin within a central plane given by $z = 0$. We have also shown that in equilibrium, without any mechanism for angular momentum or mass transportation, orbiting particles move in circles. Pringle (1981) applies similar reasoning to a continuous mass distribution to show that a disc of fluid will also remain in circular orbit in equilibrium. The mass and angular momentum distributions, however, may be changed if a process is in place that will allow so which may indeed alter the orbital motion of fluid volumes.

When defining shear stress, the reader may have been contemplating how this could aid our search for such a process to redistribute angular momentum within accretion discs. Indeed, a disc in equilibrium has *differential rotation*, that is, its angular velocity decreases as its radius increases. Molecular viscous stress therefore has the potential to cause inner parts of the disc to shear against the outer parts and redistribute their angular momentum outwards. We would then expect the decrease in angular momentum of the inner parts to force the mass within them to move to a smaller orbit in order to conserve the overall angular momentum and return to a minimum energy state; this allows for accretion. Shear stress, therefore, seems to fit our conditions perfectly.

Through the use of the equation of mass conservation and the Navier-Stokes equation of motion, we can directly model the mechanics of this process. Our method is based upon that of Frank et al. (1992, chapter 5) and Pringle (1981). For a less formal derivation, the reader should be directed to the works of Choudhuri (1998, pp. 94-102). We allow for the transport of mass by introducing an axisymmetric *radial mass inflow* with velocity v_r. Let us also impose the boundary condition that all velocity components of the fluid vanish in the limit $z \to \pm\infty$ since the gravitational forces due to the central mass will be negligible at extreme distances.

6.1 Conservation of Mass Analysis

We begin our analysis by considering the equation of mass conservation, which in cylindrical coordinates is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

using expressions for vector calculus operators given in appendix A. Integrating this over the entirety of the z and θ-coordinates to obtain an expression in r, we find

$$\begin{aligned} 0 &= \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial \rho}{\partial t}\, dz\, d\theta + \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)\, dz\, d\theta \\ &\quad + \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)\, dz\, d\theta + \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial}{\partial z}(\rho v_z)\, dz\, d\theta \\ &= \frac{\partial}{\partial t}\int_0^{2\pi}\int_{-\infty}^{+\infty} \rho\, dz\, d\theta + \frac{1}{r}\frac{\partial}{\partial r}\int_0^{2\pi}\int_{-\infty}^{+\infty} r\rho v_r\, dz\, d\theta \\ &\quad + \frac{1}{r}\int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\partial}{\partial \theta}(\rho v_\theta)\, d\theta\, dz + \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial}{\partial z}(\rho v_z)\, dz\, d\theta. \end{aligned} \tag{6.1}$$

We define the *surface density* of the disc, $\Sigma(r,t)$, as the mass per unit surface area given by

$$\Sigma = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty} \rho\, dz\, d\theta$$

measured in $\mathrm{kg} \cdot \mathrm{m}^{-2}$. The mass of an annulus between two radii, r_1 and r_2, is therefore given by

$$\int_{r_1}^{r_2} 2\pi\Sigma r\, dr. \tag{6.2}$$

Introducing this notation, equation 6.1 becomes

$$
\begin{aligned}
0 = 2\pi\frac{\partial\Sigma}{\partial t} &+ \frac{1}{r}\frac{\partial}{\partial r}\int_0^{2\pi}\int_{-\infty}^{+\infty} r\rho v_r \, dz \, d\theta \\
&+ \frac{1}{r}\int_{-\infty}^{+\infty}\left[\rho v_\theta\right]_{\theta=0}^{\theta=2\pi} dz + \int_0^{2\pi}\left[\rho v_z\right]_{-\infty}^{+\infty} d\theta.
\end{aligned}
$$

Since the disc is 2π periodic in the θ-coordinate by definition, $\rho(2\pi)v_\theta(2\pi) = \rho(0)v_\theta(0)$, and by imposing the boundary condition that $v_z \to 0$ as $z \to \pm\infty$, we see

$$
0 = 2\pi\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial\mathcal{F}}{\partial r} \tag{6.3}
$$

where $\mathcal{F}(r,t)$ is the *radial mass flux* given by

$$
\mathcal{F} = \int_0^{2\pi}\int_{-\infty}^{+\infty} r\rho v_r \, dz \, d\theta. \tag{6.4}
$$

Let us also define a *mean radial inflow*, $\bar{v}_r(r,t)$, averaged and density over the z and θ components of the disc such that

$$
\mathcal{F} = 2\pi r\bar{v}_r\Sigma. \tag{6.5}
$$

$\mathcal{F}$ is a measure of how much mass passes through an annulus at radius r at some time t given in $\mathrm{kg}\cdot\mathrm{s}^{-1}$. This allows equation (6.3) to be written as

$$
\begin{aligned}
0 &= 2\pi\frac{\partial\Sigma}{\partial t} + \frac{2\pi}{r}\frac{\partial}{\partial r}(r\bar{v}_r\Sigma) \\
\Rightarrow 0 &= \frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}_r\Sigma),
\end{aligned} \tag{6.6}
$$

which expresses the conservation of mass of the accretion disc.

6.2 Equation of Motion Analysis

Our attention is now turned towards the Navier Stokes equation of motion, for which we choose the body forces, $\mathbf{F}$, to be gravitational forces due to the central mass, written $\mathbf{F} = \nabla\varphi$ where φ is the gravitational potential defined in equation (2.11). The equation of motion therefore becomes

$$
\rho\frac{D\mathbf{v}}{Dt} = -\rho\nabla\varphi - \nabla p + \nabla\cdot\mathbf{T}. \tag{6.7}
$$

For now, we will focus on the θ-component of this equation as this is where the shear stress will make the greatest influence. In cylindrical coordinates, the θ-component is given by

$$
\rho\left(r\frac{Dv_\theta}{Dt} + v_r v_\theta\right) = -\frac{\partial p}{\partial\theta} + \frac{\partial}{\partial r}(rT_{\theta r}) + \frac{\partial}{\partial\theta}(T_{\theta\theta}) + \frac{\partial}{\partial z}(rT_{\theta z})
$$

where we have multiplied through by r and grouped the derivatives with respect to θ. Now, expanding the Lagrangian derivative using appendix A, we see

$$\begin{aligned} r\frac{Dv_\theta}{Dt} + v_r v_\theta &= r\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z}\right) + v_r v_\theta \\ &= \frac{\partial}{\partial t}(rv_\theta) + \left(rv_r\frac{\partial v_\theta}{\partial r} + v_r v_\theta\right) + \frac{v_\theta}{r}\frac{\partial}{\partial \theta}(rv_\theta) + v_z\frac{\partial}{\partial z}(rv_\theta) \\ &= \frac{\partial h}{\partial t} + v_r\frac{\partial h}{\partial r} + \frac{v_\theta}{r}\frac{\partial h}{\partial \theta} + v_z\frac{\partial h}{\partial z} \\ &= \frac{Dh}{Dt} \end{aligned}$$

where we have reintroduced $h = r^2\Omega$ as the specific orbital angular momentum. The θ-component of the equation of motion therefore becomes

$$\rho\frac{Dh}{Dt} = -\frac{\partial p}{\partial \theta} + \frac{\partial}{\partial r}(rT_{\theta r}) + \frac{\partial}{\partial \theta}(T_{\theta\theta}) + \frac{\partial}{\partial z}(rT_{\theta z}). \tag{6.8}$$

Let us now assume that $\Omega = \Omega(r) \Rightarrow h = h(r)$. By our analysis in section 3, this does not seem unreasonable as we have shown this holds for particle motion if we assume the gravitational field is fixed. We will discuss its validity in the case of a continuous disc in due course. Equation (6.8) therefore becomes

$$\rho v_r\frac{dh}{\partial r} = \frac{\partial}{\partial r}(rT_{\theta r}) + \frac{\partial}{\partial \theta}(t_{\theta\theta} - p) + \frac{\partial}{\partial z}(rT_{\theta z}).$$

Integrating this with respect to θ and z across the whole disc gives

$$\begin{aligned} \int_0^{2\pi}\int_{-\infty}^{+\infty} \rho v_r\frac{dh}{\partial r}\,dz\,d\theta &= \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial}{\partial r}(rT_{\theta r})\,dz\,d\theta \\ &+ \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial}{\partial \theta}(T_{\theta\theta} - p)\,dz\,d\theta + \int_0^{2\pi}\int_{-\infty}^{+\infty} \frac{\partial}{\partial z}(rT_{\theta z})\,dz\,d\theta \end{aligned}$$

and, by imposing our boundary conditions and periodicity, this becomes

$$\Rightarrow \frac{dh}{dr}\frac{\mathcal{F}}{r} = \frac{\partial}{\partial r}\int_0^{2\pi}\int_{-\infty}^{+\infty}(rT_{\theta r})\,dz\,d\theta$$

where $\mathcal{F}$ is the radial mass flux. Multiplying through by r gives us

$$\frac{dh}{dr}\mathcal{F} = -\frac{\partial \mathcal{G}}{\partial r} \tag{6.9}$$

where $\mathcal{G} = -\int_0^{2\pi}\int_{-\infty}^{+\infty}(r^2T_{\theta r})\,dz\,d\theta$. Given two thin annuli on either side of a radius r, $\mathcal{G}$ is the *viscous torque* exerted on the outer annulus by the inner annulus measured in newton metres ($\mathrm{N}\cdot\mathrm{m}$). From our expression of the deviatoric stress tensor in equation (5.12)

$$T_{\theta r} = \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) = \mu\left[\frac{d}{dr}(r\Omega) - \Omega\right] = \mu r\frac{d\Omega}{dr},$$

since u_r is an axisymmetric mass inflow. The viscous torque is therefore given by

$$\mathcal{G} = -r^3 \frac{d\Omega}{dr} \int_0^{2\pi} \int_{-\infty}^{+\infty} \mu \, dz \, d\theta = -r^3 \frac{d\Omega}{dr} \int_0^{2\pi} \int_{-\infty}^{+\infty} \rho\nu \, dz \, d\theta,$$

where ν is the kinematic viscosity defined in equation (5.11). Let us define $\bar{\nu}(r,t)$ as the mean kinematic viscosity averaged across the z and θ components of the disc such that

$$\mathcal{G} = -r^3 \frac{d\Omega}{dr} \bar{\nu} 2\pi\Sigma \tag{6.10}$$

then equation (6.9) becomes,

$$\begin{aligned} \frac{dh}{dr}\mathcal{F} &= -\frac{\partial}{\partial r}\left(-r^3 \frac{d\Omega}{dr}\bar{\nu}2\pi\Sigma\right) \\ \Rightarrow \frac{dh}{dr}\bar{v}_r\Sigma &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^3 \frac{d\Omega}{dr}\bar{\nu}\Sigma\right), \end{aligned} \tag{6.11}$$

by substituting our expression for $\mathcal{F}$ from equation (6.4). This describes the conservation of angular momentum within the disc. We note that if Ω decreases with radius then the viscous torque is positive which suggests that angular momentum is indeed transported radially outwards since torque is a measure of how much a force causes rotation.

6.3 Derivation of the Surface Density Diffusion Equation

We now combine the two equations from our analysis describing the behaviour of the accretion disc, the equations for conservation of mass and angular momentum. In an attempt to eliminate $\bar{v}_r$, equation (6.6) tells us

$$\frac{\partial}{\partial r}(r\Sigma\bar{v}_r) = -r\frac{\partial \Sigma}{\partial t}$$

and through rearranging equation (6.11) and differentiating with respect to r we find

$$\frac{\partial}{\partial r}\left(r\Sigma\bar{v}_r\right) = \frac{\partial}{\partial r}\left[\left(\frac{\partial h}{\partial r}\right)^{-1} \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3 \frac{d\Omega}{dr}\right)\right],$$

therefore, we obtain the differential equation

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{dh}{dr}\right)^{-1} \frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^3 \frac{d\Omega}{dr}\right)\right]. \tag{6.12}$$

At this stage, we simplify our problem by specifying the central mass to be a point particle and assuming that the orbiting matter within the disc follows coplanar, Keplerian orbits. Again, we discuss the validity of this assumption in a later section. We can therefore make use of the results from section 3, specifically equations (3.7)-(3.9), to find

$$\frac{d\Omega}{dr} = -\frac{3}{2}\sqrt{GM}r^{-\frac{5}{2}}, \qquad \frac{dh}{dr} = \frac{1}{2}\sqrt{GM}r^{-\frac{1}{2}}.$$

Substituting these expressions into our differential equation (6.12) and simplifying gives the *surface density diffusion equation*

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{\frac{1}{2}}\frac{\partial}{\partial r}\left(\bar{\nu}\Sigma r^{\frac{1}{2}}\right)\right]. \tag{6.13}$$

which can be used to analyse the time evolution of a Keplerian accretion disc. Given a solution for this diffusion equation, (6.11) can be used to find $\bar{v}_r$ given by

$$\bar{v}_r = -\frac{3}{r^{1/2}\Sigma}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\bar{\nu}\Sigma\right) \tag{6.14}$$

which can in turn provide an insight into the accretion rates that arise due to the transport of angular momentum from shear stress.

6.4 Analysis of the Diffusion Equation

In order to obtain some qualitative results from this analysis, it becomes imperative that we make some assumption on the quantity $\bar{\nu}$; we will discuss the case where $\bar{\nu}$ is a constant. It is important to note that this is not likely to be true, but our aim for now is to analyse the general mechanics of the disc; for this purpose it will suffice. The diffusion equation can therefore be written

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} &= \frac{3\bar{\nu}}{r}\frac{\partial}{\partial r}\left[r^{\frac{1}{2}}\frac{\partial}{\partial r}\left(\Sigma r^{\frac{1}{2}}\right)\right] \\ \Rightarrow r^{\frac{1}{2}}\frac{\partial \Sigma}{\partial t} &= \frac{3\bar{\nu}}{r}\left[r^{\frac{1}{2}}\frac{\partial}{\partial r}\left(r^{\frac{1}{2}}\right)\frac{\partial}{\partial r}\left(\Sigma r^{\frac{1}{2}}\right)\right] \\ \Rightarrow \frac{\partial}{\partial t}\left(r^{\frac{1}{2}}\Sigma\right) &= \frac{3\bar{\nu}}{r}\left(r^{\frac{1}{2}}\frac{\partial}{\partial r}\right)^2\left(\Sigma r^{\frac{1}{2}}\right), \end{aligned}$$

which we continue to solve in detail. Defining $s = 2r^{\frac{1}{2}}$ such that $\frac{\partial}{\partial s} = r^{\frac{1}{2}}\frac{\partial}{\partial r}$, a change of variables simplifies this equation to

$$\frac{\partial}{\partial t}\left(r^{\frac{1}{2}}\Sigma\right) = \frac{12\bar{\nu}}{s^2}\frac{\partial^2}{\partial s^2}\left(r^{\frac{1}{2}}\Sigma\right) \tag{6.15}$$

which can be seen as a separable partial differential equation for $r^{\frac{1}{2}}\Sigma$. We seek functions $T(t)$ and $S(s)$ such that $r^{\frac{1}{2}}\Sigma = T(t)S(s)$, giving

$$S\frac{dT}{dt} = \frac{12\bar{\nu}}{s^2}T\frac{d^2S}{ds^2}$$

$$\Rightarrow \frac{1}{t}\frac{dT}{dt} = \frac{12\bar{\nu}}{s^2}\frac{d^2S}{ds^2}\frac{1}{S} = c \tag{6.16}$$

for some constant c. The partial differential equation is reduced to two ordinary differential equations in t and s. In order to solve them, we must first derive a further boundary condition on the disc.

Observations have shown that at some radius $r = r_{in}$ towards the inner boundary where the disc meets the accreting object, there is a rapid transition from the angular velocity of the disc to that of the central mass at a point where the viscous torque $\mathcal{G}$ vanishes. This seems logical since we would not expect the accretion disc to draw angular momentum from the accreting object itself. For a point particle central mass we can take $r_{in} \to 0$ and therefore our boundary condition states that $\mathcal{G} \to 0$ as $r \to 0$. Recalling our assumption of Keplerian orbit in the disc, we see $\mathcal{G} = -r^{\frac{1}{2}}(GM)^{\frac{1}{2}}\bar{\nu}2\pi\Sigma$, therefore our boundary condition imposes that $r^{\frac{1}{2}}\Sigma \to 0$ as $r \to 0$.

With this in mind, let us continue by focussing on the ODE for S and attempt to obtain a solution in terms of r. Changing our variable to r at this stage will result in a differential equation in a common form that we can solve. We see

$$\frac{d^2S}{ds^2} = \frac{d}{ds}\left(\frac{dS}{dr}\frac{dr}{ds}\right) = \frac{d}{dr}\left(\frac{dS}{dr}r^{\frac{1}{2}}\right)r^{\frac{1}{2}} = \frac{d^2S}{dr^2}r + \frac{1}{2}\frac{dS}{dr},$$

therefore, the equation for S can be written

$$\frac{12\bar{\nu}}{4r}\left(\frac{d^2S}{dr^2}r + \frac{1}{2}\frac{dS}{dr}\right)\frac{1}{S} = c$$

$$\Rightarrow 12\hat{\nu}\frac{d^2S}{dr^2}r + 6\hat{\nu}\frac{dS}{dr} - 4rSc = 0. \tag{6.17}$$

Let us now assume that $S(r) = r^{\alpha}P(r)$ for some arbitrary α. Substituting this in to equation (6.17) and simplifying gives

$$0 = r^2\frac{d^2P}{dr^2} + \frac{dP}{dr}\left(2\alpha + \frac{1}{2}\right) + \left(\alpha^2 - \frac{\alpha}{2}\right)P - \frac{c}{r\bar{\nu}}r^2P.$$

Moreover, by choosing $\alpha = \frac{1}{4}$, this conveniently reduces to the fourth order Bessel differential equation

$$0 = r^2\frac{d^2P}{dr^2} + \frac{dP}{dr}r + \left(k^2r^2 - \frac{1}{16}\right)P$$

where $k^2 = -\frac{c}{3\hat{\nu}}$. The general solution for P is therefore given by

$$P(r) = AJ_{1/4}(kr) + BY_{1/4}(kr)$$

for some constants A and B and subsequently

$$S(r) = r^{\frac{1}{4}} \left(AJ_{1/4}(kr) + BY_{1/4}(kr) \right).$$

Imposing our inner boundary condition, it must follow that $S \to 0$ as $r \to 0$ and since $r^{\frac{1}{4}} Y_{1/4}(kr) \nrightarrow 0$ we find $B = 0$. From our expression for the constant k^2, we can deduce that $c = -3k^2\bar{\nu}$. Our differential equation for T from equation (6.16) therefore becomes

$$\frac{1}{T}\frac{dT}{dt} = -k^2 3\bar{\nu} \Rightarrow T = Ce^{-k^2 3\bar{\nu}t}$$

for a constant C. Therefore, $\Sigma(r,t) \propto r^{-\frac{1}{4}} J_{1/4}(kr) e^{-3\bar{\nu}k^2 t}$ and the general solution for the surface density of the accretion disc is

$$\Sigma(r,t) = \int_0^\infty f(k) r^{-\frac{1}{4}} J_{1/4}(kr) e^{-3\bar{\nu}k^2 t}\, dk \tag{6.18}$$

for some function $f(k)$ satisfying

$$\Sigma(r,0) = \int_0^\infty f(k) r^{-\frac{1}{4}} J_{1/4}(kr)\, dk$$

$$\Rightarrow r^{\frac{1}{4}} \Sigma(r,0) = \int_0^\infty \left(\frac{f(k)}{k} \right) J_{1/4}(kr) k\, dk. \tag{6.19}$$

In order to find $f(k)$, we make use of the *Hankel transform* of order $\frac{1}{4}$ given by Debnath and Bhatta (2006) as

$$H(k) = \int_0^\infty h(r) J_n(kr) r\, dr,$$

$$h(r) = \int_0^\infty H(k) J_n(kr) k\, dk,$$

for functions $H(k)$ and $h(r)$. Applying this transform to (6.19), it follows that

$$\frac{f(k)}{k} = \int_0^\infty r^{\frac{1}{4}} \Sigma(r,0) J_{1/4}(kr) r\, dr$$

$$\Rightarrow f(k) = k \int_0^\infty \Sigma(r,0) J_{1/4}(kr) r^{\frac{5}{4}}\, dr,$$

and substituting this in to equation (6.18) gives

$$\Sigma(r,t) = \int_0^\infty k \left[\int_0^\infty \Sigma(q,0) J_{1/4}(kq) q^{\frac{5}{4}}\, dq \right] r^{-\frac{1}{4}} J_{1/4}(kr) e^{-3\bar{\nu}k^2 t}\, dk$$

where q is a dummy variable and $\Sigma(q,0)$ is the initial surface density. The integrals can be swapped and rearranged to give

$$\Sigma(r,q,t) = \int_0^\infty \Sigma(q,0) \Gamma(r,q,t)\, dq$$

where $\Gamma(r, q, t)$ is given by

$$\Gamma(r, q, t) = q^{\frac{5}{4}} r^{-\frac{1}{4}} \int_0^\infty J_{1/4}(kq) J_{1/4}(kr) k e^{-3\bar{\nu} k^2 t} \, dk, \tag{6.20}$$

to which we now turn our attention to. From the results contained in the handbook of Olver et al. (2010, equation 10.22.67) we see

$$\int_0^\infty J_n(kq) J_n(kr) k e^{-p^2 k^2} \, dk = \frac{1}{2p^2} \exp\left(-\frac{q^2 + r^2}{4p^2}\right) I_n\left(\frac{qr}{2p^2}\right)$$

for $\Re(n) > -1$ and $\Re(p^2) > 0$ where I_n is the modified Bessel function of order n. Applying this to equation (6.20) with $p^2 = 3\bar{\nu} t > 0$ and $n = \frac{1}{4}$ gives

$$\Gamma(r, q, t) = q^{\frac{5}{4}} r^{-\frac{1}{4}} \exp\left(-\frac{q^2 + r^2}{12\bar{\nu}} t\right) I_{1/4}\left(\frac{qr}{6\bar{\nu} t}\right)$$

and therefore the solution to the surface density diffusion equation is given by

$$\Sigma(r, q, t) = \int_0^\infty \Sigma(q, 0) q^{\frac{5}{4}} r^{-\frac{1}{4}} \frac{1}{6\bar{\nu} t} \exp\left(-\frac{q^2 + r^2}{12\bar{\nu} t}\right) I_{1/4}\left(\frac{qr}{6\bar{\nu} t}\right) ds.$$

Following Frank et al. (1992, p. 69) we find the *Green's function*, defined as the solution for $\Sigma(r, q, t)$ taking the initial surface density distribution as that of a ring of mass m_0 at some radius $q = r_0$. Since the mass of a disc between two radii satisfies equation (6.2), it follows that this initial surface density is given by

$$\Sigma(q, 0) = \frac{m}{2\pi r_0} \delta(s - r_0)$$

giving the Green's function to be

$$\Sigma(\tau, x) = \frac{m}{\pi r_0^2} \tau^{-1} x^{-\frac{1}{4}} \exp\left(-\frac{1 + x^2}{\tau}\right) I_{1/4}\left(\frac{2x}{\tau}\right) \tag{6.21}$$

where we have introduced the dimensionless parameters $x = \frac{r}{r_0}$ and $\tau = \frac{12\bar{\nu} t}{r_0^2}$. A plot of the solution at different scaled times τ is shown in figure 7.

6.5 Discussion of Solution and Steady State Disc

We have shown that viscosity has the effect of spreading out a thin disc placed in Keplerian orbit. Initially, the surface density appears normally distributed about $r = r_0$. As time evolves, the majority of mass drifts towards the centre, while a small amount of matter moves out to larger radii. When $\bar{\nu}$ is constant, from equation (6.14) we see $\bar{v}_r \sim \frac{\bar{\nu}}{r}$. In fact, we can write

$$\bar{v}_r = -3\bar{\nu} \frac{\partial}{\partial r} \left[\ln\left(r^{\frac{1}{2}} \Sigma\right)\right] = -\frac{3\bar{\nu}}{r_0} \frac{\partial}{\partial x} \left[\ln\left(x^{\frac{1}{2}} \Sigma\right)\right]$$

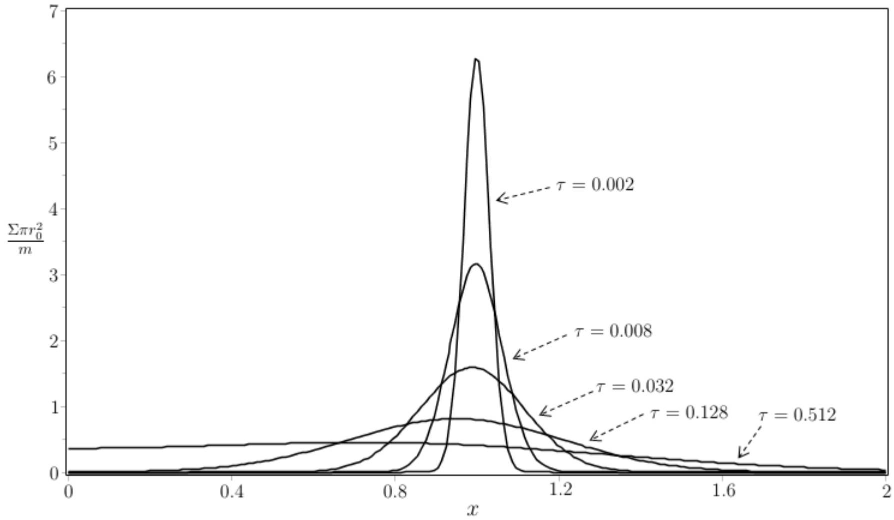

Figure 7: The spreading of a ring of mass m_1 placed in Keplerian orbit at radius r_0 around a central mass due to viscosity. Adapted from Pringle (1981).

and, substituting from (6.21), this gives

$$\bar{v}_r = -\frac{3\bar{\nu}}{r_0}\frac{\partial}{\partial x}\left[\frac{1}{4}\ln x - \frac{1+x^2}{\tau} + \ln I_{1/4}\left(\frac{2x}{\tau}\right)\right]$$

where we have dropped the constant terms as they will disappear due to the differential. For $2x \gg \tau$, Frank et al. (1992, p. 70) states that $I_{1/4}\left(\frac{2x}{\tau}\right) \propto \left(\frac{\tau}{2x}\right)^{1/2} \exp\left(\frac{2x}{\tau}\right)$ and for $2x \ll \tau$, $I_{1/4}\left(\frac{2x}{\tau}\right) \propto \left(\frac{2x}{\tau}\right)^{1/4}$. Thus, when $2x \gg \tau$

$$\begin{aligned}\bar{v}_r &\sim -\frac{3\bar{\nu}}{r_0}\frac{\partial}{\partial x}\left[\frac{1}{4}\ln x - \frac{1+x^2}{\tau} + \frac{1}{2}\ln\left(\frac{\tau}{2x}\right) + \frac{2x}{\tau}\right] \\ &= \frac{3\bar{\nu}}{r_0}\left(\frac{1}{4x} + \frac{2x}{\tau} - \frac{2}{\tau}\right) > 0,\end{aligned}$$

and when $2x \ll \tau$

$$\begin{aligned}\bar{v}_r &\sim -\frac{3\bar{\nu}}{r_0}\frac{\partial}{\partial x}\left[\frac{1}{4}\ln x - \frac{1+x^2}{\tau} + \frac{1}{4}\ln(2x) - \frac{1}{4}\ln\tau\right] \\ &= -\frac{3\bar{\nu}}{r_0}\left(\frac{1}{2x} - \frac{2x}{\tau}\right) < 0.\end{aligned}$$

We therefore find that the outer parts of the disc move further outwards, carrying with them the angular momentum of the inner parts. The inner parts are then forced to take a smaller orbit eventually accreting onto the central mass. This is in direct agreement with our original hypothesis stated when introducing this section. Moreover, we see the radius at which v_r changes sign actually increases with time, meaning that mass that has originally extended to a further radius will at some point in the future be forced inwards to accrete. The limit as τ increases see's a system with almost the entire initial mass accreted, while the angular momentum is all carried by minimal mass to extremely large radii.

From figure 7 we can estimate the timescale, t_{visc}, on which viscosity initially spreads the original ring out by, through analysing the different widths of the annulus of mass, σ, at different times given in table 1.

Table 1: Timescale Analysis

$\approx \sigma$	τ	$\frac{\approx\sigma^2}{\tau}$
0.4	0.002	80
0.8	0.008	80
1.5	0.032	70.3

We have named the width of the annulus σ since it roughly corresponds to the standard deviation of the distribution of mass at the time τ. Our observations seem to suggest that the radii of the disc increases such that $\sigma^2\tau^{-1} \sim 1$. We can therefore deduce that the viscosity spreads mass on a timescale τ_{visc} where $x^2\tau_{\text{visc}}^{-1} \sim 1$, or equivalently $\frac{r^2}{\bar{\nu}t_{\text{visc}}} \sim 1$, giving

$$t_{\text{visc}} \sim \frac{r^2}{\bar{\nu}}. \tag{6.22}$$

Since this subsequently implies that $t_{\text{visc}} \sim \frac{r}{\bar{v}_r}$, it is also known as the *radial drift timescale* as it estimates the timescale on which a disc annulus moves a radial distance r. In general, Frank et al. (1992) states that the external conditions of an accretion disc change on timescales much longer than t_{visc}. Moreover, it is also common that the disc is fed by mass from surrounding matter or a companion star in a binary system such that the mass lost towards the inner boundary is replenished. The system will then tend towards a state of equilibrium and can be modelled as a steady state by equating any time derivatives to zero; this approach is commonly used in accretion disc research.

6.6 Keplerian Assumption Validation

Following on from our conclusion that the disc can be considered steady, we are now in a position where our assumption of Keplerian velocity can be analysed. We firstly consider the vertical component of the equation of motion (6.7) given by

$$\rho\frac{\partial u_z}{\partial t}+\rho(\mathbf{v}\cdot\nabla)v_z=-\rho\frac{\partial\varphi}{\partial z}-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rT_{rz})+\frac{1}{r}\frac{\partial}{\partial\theta}(T_{\theta z})+\frac{\partial}{\partial z}(T_z z),$$

however, we do not expect any vertical stresses and only minimal vertical flow. The dominating terms therefore give

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-\frac{\partial\varphi}{\partial z}.$$

This situation, where gravity is balanced by pressure, is known as *vertical hydrostatic equilibrium.* Through substituting our expression for the gravitational potential given in equation (3.4), this becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=GM\frac{\partial}{\partial z}\left((r^2+z^2)^{-\frac{1}{2}}\right). \tag{6.23}$$

For a thin disc we expect $z \ll r$, so expanding $(r^2+z^2)^{-\frac{1}{2}}$ as a Taylor series about $z=0$ gives

$$(r^2+z^2)^{-\frac{1}{2}}=\frac{1}{r}-\frac{rz^2}{2r^4}+o(z^2) \tag{6.24}$$

$$\Rightarrow \frac{\partial}{\partial z}\left((r^2+z^2)^{-\frac{1}{2}}\right)\approx-\frac{z}{r^3}$$

and equation (6.23) becomes, to a first approximation,

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-\frac{GMz}{r^3}=-\Omega_k^2 z \tag{6.25}$$

where Ω_k is the Keplerian angular velocity. Let us define H as a typical scale-height for the discs in the z-direction giving $\frac{\partial p}{\partial z}\sim\frac{p}{H}$. So, by a comparison of magnitudes, we find

$$\frac{1}{\rho}\frac{p}{H}\approx\frac{GMH}{r^3}\Rightarrow c_s r\approx\left(\frac{GM}{r}\right)^{\frac{1}{2}}H$$

where $c_s^2=\frac{p}{\rho}$ is defined as the *isothermal sound speed.* Therefore, for our thin assumption to hold, we must have

$$c_s \ll \left(\frac{GM}{r}\right)^{\frac{1}{2}}. \tag{6.26}$$

We now turn to the radial component of the equation of motion, given by

$$\rho\left(\frac{\partial v_r}{\partial t}+\mathbf{v}\cdot\nabla v_r-\frac{v_\theta^2}{r}\right)=-\rho\frac{\partial\varphi}{\partial r}-\frac{\partial p}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr})+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta}+\frac{\partial t_{rz}}{\partial z}-\frac{T_{\theta\theta}}{r}.$$

Again, we expect T_{rr}, $T_{\theta\theta}$ and T_{rz} to be negligible and for there to be minimal variation on v_r and $T_{r\theta}$ in the θ and z directions. Therefore, for a steady disc, the dominating terms give

$$\rho\left(v_r\frac{\partial v_r}{\partial r}-\frac{v_\theta^2}{r}\right)=-\rho\frac{\partial\varphi}{\partial r}-\frac{\partial p}{\partial r}. \tag{6.27}$$

In the thin disc approximation, we make the estimation $\frac{\partial\varphi}{\partial r}\approx\frac{GM}{r^2}$ using the taylor expansion (6.24). Equation (6.27) therefore becomes

$$v_r\frac{\partial v_r}{\partial r}-\frac{v_\theta^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{GM}{r^2}=0.$$

Let us firstly compare the scales of the pressure term with the gravity term. We find

$$\frac{1}{\rho}\frac{\partial p}{\partial r}\sim\frac{c_s^2}{r}$$

and by equation (6.26)

$$\frac{c_s^2}{r}\ll\frac{GM}{r^2}$$

so the pressure term is negligible in comparison with the gravity term. We now evaluate the scaling of $v_r\frac{\partial v_r}{\partial r}$. We know $v_r\sim\frac{\bar\nu}{r}\sim\frac{\bar\mu}{\rho r}$ where we have defined $\bar\mu$ as an averaged dynamic viscosity, and by the definition of μ, $[\mu]=\mathrm{ML^{-1}T^{-1}}$. We also see

$$\left[\frac{p}{\Omega_k}\right]=\frac{\mathrm{ML^{-1}T^{-2}}}{\mathrm{T^{-1}}}=\mathrm{ML^{-1}T^{-1}}$$

so we can write

$$\bar\mu=\frac{\alpha p}{\Omega_k}$$

for some dimensionless parameter α, which can be shown to be between zero and approximately one. This relation is called the *alpha viscosity prescription* and was first adopted in the work by Shakura and Sunyaev (1973) on steady accretion discs. It has been extensively used to parameterise forms of different stress that may occur in accretion discs. On this occasion, we use it simply for a scaling analysis. From equation (6.25) we find $\Omega_k\sim\frac{c_s}{H}$, therefore,

$$\bar\mu\sim\frac{\alpha pH}{c_s}\Rightarrow v_r\sim\frac{\alpha PH}{c_s\rho r}=\frac{\alpha c_s H}{r}$$

and since $H\ll r$ it must follow that $v_r\ll c_s$; the radial velocity must be highly subsonic. As a side note, we state that the alpha parameterisation leads to $\bar\nu\sim\alpha c_s H$. Therefore,

$$v_r\frac{\partial v_r}{\partial r}\sim\frac{v_r^2}{r}\ll\frac{c_s^2}{r},$$

so this term is even smaller than the pressure term. To a leading order of magnitude, the radial component of the equation of motion becomes

$$-\frac{v_\theta^2}{r} + \frac{GM}{r^2} = 0$$

and, by defining the *Mach number*, $\mathcal{M}$, according to Frank et al. (1992) such that $\mathcal{M} = \frac{v_\theta}{c_s}$, we see the azimuthal velocity satisfies

$$v_\theta = \left(\frac{GM}{r}\right)^{\frac{1}{2}} \left[1 + O(\mathcal{M}^{-2})\right].$$

We can therefore conclude that our assumption of Keplerian angular velocity appears to be very reasonable. Furthermore, our thin disc assumption given in (6.26) implies that the circular velocity must be highly supersonic.

6.7 Accretion Rates and Luminosities of a Steady Disc

In order to understand the efficiency of our model and analyse whether it fits observed or expected results, we now analyse the rate of accretion and energy release for the disc. Under the steady conditions described in the previous section, the equation of mass conservation (6.6) becomes

$$\frac{\partial}{\partial r}(r\bar{v}_r\Sigma) = 0$$

which implies that $r\bar{v}_r\Sigma$ is constant at every radius r. Consequently, the radial mass flux, $\mathcal{F}$, defined in equation (6.3) must also be constant with changing radius, so

$$\dot{M} = -\mathcal{F} = \text{constant}$$

where $\dot{M}$ is defined as the *accretion rate* with units $\text{kg} \cdot \text{s}^{-1}$. Since our Keplerian assumption has now been validated, we are safe in assuming $h = h(r)$ and $\mathcal{G} = \mathcal{G}(r)$ where $\mathcal{G}$ is the viscous torque given in equation (6.10). Equation (6.9) can therefore be written

$$\dot{M}\frac{dh}{dr} = -\frac{d\mathcal{G}}{dr}$$

which we integrate with respect to r to give

$$\dot{M}h = -\mathcal{G} + c \Rightarrow\ \mathcal{G} = \dot{M}(h - h_{in}),$$

where we have defined $h_{in} = h(r_{in})$ and found the constant c by imposing the inner boundary condition that $\mathcal{G}(r_{in}) = 0$. For a Keplerian disc

$$G = 3\pi r^{\frac{1}{2}}(GM)^{\frac{1}{2}}\bar{\nu}\Sigma, \qquad h_{in} = (GM)^{\frac{1}{2}} r_{in}^{\frac{1}{2}}, \qquad h = (GM)^{\frac{1}{2}} r^{\frac{1}{2}},$$

therefore,

$$3\pi r^{\frac{1}{2}}\bar{\nu}\Sigma = \dot{M}\left(r^{\frac{1}{2}} - r_{in}^{\frac{1}{2}}\right)$$

$$\Rightarrow \bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \left(\frac{r_{in}}{r}\right)^{\frac{1}{2}}\right). \tag{6.28}$$

This highlights that for $r \gg r_{in}$, $\Sigma \approx \frac{\dot{M}}{3\pi\bar{\nu}}$, while the mass inflow velocity, in accordance with (6.14), satisfies

$$\bar{u}_r \approx -\frac{3\bar{\nu}}{r^{1/2}}\frac{d}{dr}\left(r^{\frac{1}{2}}\right) = -\frac{3\bar{\nu}}{2r}. \tag{6.29}$$

Pringle (1981) defines the standard *dissipation rate* per unit area per unit time from fluid dynamics due to a kinematic viscosity as

$$D(r) = \frac{1}{2}\bar{\nu}\Sigma\left(r\frac{d\Omega}{dr}\right)^2.$$

By substituting in equation (6.28), the dissipation rate for our steady Keplerian disc is thus given by

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3}\left[1 - \left(\frac{r_{in}}{r}\right)^{\frac{1}{2}}\right].$$

We note that this expression does not explicitly contain $\bar{\nu}$, however, we remind the reader that it is contained in the accretion rate. The total amount of energy released due to the inflow of mass from viscosity within the disc at time t is therefore found by integrating this over r. Such a quantity is defined as the *disc luminosity*, denoted L_{disc}, and is given by Frank et al. (1992) as

$$\begin{aligned} L_{\text{disc}} &= 2\int_{r_{in}}^{\infty} D(r) 2\pi r \, dr \\ &= \frac{3GM\dot{M}}{2}\int_{r_{\text{in}}}^{\infty}\frac{1}{r^2}\left(1 - \left(\frac{r_{\text{in}}}{r}\right)^{\frac{1}{2}}\right) dr. \end{aligned}$$

Letting $y = \frac{r_{\text{in}}}{r}$, this becomes

$$L_{\text{disc}} = -\frac{3GM\dot{M}}{2r_{\text{in}}}\int_1^0 (1 - y^{\frac{1}{2}})\, dy = \frac{GM\dot{M}}{2r_{\text{in}}}.$$

This is only half of the total potential energy associated with accretion, $L_{\text{acc}} = \frac{GM\dot{M}}{r_{\text{in}}}$. The remaining half is retained by matter towards the inner boundary in the form of kinetic energy and is dissipated as it makes the transition from this boundary on to the central mass.

6.8 Confrontation with Observations

In order to analyse how our viscous model compares to observations, we must estimate a typical value for the kinematic viscosity $\bar{\nu}$, which we have assumed to be a molecular shear stress. From section 5.4, we remind ourselves that $\bar{\nu} \sim \lambda v_{\text{mol}}$ where v_{mol} and λ are the mean speed and distance that a free molecule travels before colliding. We take v_{mol} to be the typical isothermal speed of sound, c_s, in a gas of temperature T given by Frank et al. (1992, p. 13) as

$$c_s \approx \left(\frac{T}{10^4}\right)^{\frac{1}{2}} \times 10\,\text{km}\cdot\text{s}^{-1}$$

where T is measured in kelvin (K). The mean free path is also given by

$$\lambda \approx \frac{7\times 10^5}{\ln\Lambda}\frac{T^2}{N}\,\text{cm}$$

where $\ln\Lambda$ is a constant no less than 10 for astrophysical fluids and N is the *gas density* (cm^{-3}). Taking $\ln\Lambda = 10$, $T \approx 10^4\,\text{K}$ and $N \approx 10^{15}\,\text{cm}^{-3}$ as typical values of an accretion disc yields $v_{\text{mol}} \approx 10^4\,\text{m}\cdot\text{s}^{-1}$, $\lambda \approx 7\times 10^{-5}\,\text{m}$ which subsequently gives $\bar{\nu} \approx 0.7\,\text{m}^2\cdot\text{s}^{-1}$. At a typical distance of $r \approx 10^8\,\text{m}$ from the central mass, the viscous timescale from equation (6.22) is therefore estimated by $t_{visc} \approx 10^{16}\,\text{s} \approx 3\times 10^8$ years. The mass inflow is approximated using (6.29) as $v_r \approx -10^{-8}\,\text{m}\cdot\text{s}^{-1} \approx -30\,\text{cm}\cdot\text{year}^{-1}$. Since the current age of the universe is approximately 1.3×10^{10} years, the viscous timescale and mass inflow due to molecular viscosity does not seem feasible. Given a typical central body of mass $M = 1\,\text{M}_\odot$, the *Reynolds number*, Re, can be used as a ratio of the inertial forces to the viscous forces, given by

$$\text{Re} = \frac{u_\theta r}{\bar{\nu}} = \frac{\sqrt{GMr}}{\bar{\nu}} \approx 10^{11},$$

therefore, the viscous terms associated with molecular shear stress are shown to be entirely negligible.

In order to explain the observed luminosities of accretion discs, an extremely larger viscosity must be present. This lead to research based on general parameterisation of viscous stress using the alpha viscosity prescription by the likes of Shakura and Sunyaev (1973) in order to deduce a mechanism that would allow the extreme levels of stress to exist. It became accepted that a turbulent process must be in place. A beta parameterisation was also introduced by Piran (1978) where the kinematic viscosity was assumed proportional to the gas pressure. It was concluded that the *Rayleigh stability criterion* $\frac{d}{dr}\left(r^2\Omega\right) > 0$ must be broken in order for a linear instability to exist and create turbulence, that is, the specific orbital angular momentum <u>must decrease with radius</u>. Indeed, this goes against the foundations of a Keplerian disc. An efficient process for this could not be deduced within the subject of hydrodynamics, urging discussions around the introduction of an electrically conducting accretion disc with a magnetic field. We therefore continue this project by turning to the subject of magnetohydrodynamics (MHD). This will allow us to give a brief insight into an effective candidate for the process attributable to the mechanics of accretion discs.

7 Magnetohydrodynamics Equations

7.1 Introduction to MHD

We begin our insight into the magnetohydrodynamical properties of electrically conducting accretion discs by introducing some key concepts of the subject. Consider a continuum of freely moving particles, that is, a fluid with some velocity field $\mathbf{v}$. If the fluid is electrically conducting, each particle has some *electric charge*, measured in coulombs (C) such that $\mathrm{C} = \mathrm{A} \cdot \mathrm{s}$ where A represents amps, the SI unit of electric current. Such a fluid is called a plasma. In a similar approach to our definition of density in section 5, we define the *charge density* ($\mathrm{C} \cdot \mathrm{m}^{-3}$) of a volume element δV centred at the fixed point P in a plasma as

$$\rho^*(\mathbf{r}, t) = \frac{\text{electric charge in } \delta V}{\delta V}.$$

The total charge, therefore, of a plasma contained in a volume V is given by

$$Q = \int_V \rho^* \, dV.$$

The corresponding *electric current*, I, through a surface S is given by the rate that charge passes through the surface, i.e. the flux of ρ^* through S, and the *current density*, $\mathbf{j}$, is the value of I per unit area measured in $\mathrm{A} \cdot \mathrm{m}^{-2}$. It therefore follows that

$$\mathbf{j} = \rho^* \mathbf{v}.$$

The force per unit mass upon a plasma due to this electric charge, $\mathbf{F_e}$, is a function of both the charge and current density and is parameterised by the vectors $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ such that

$$\mathbf{F_e} = \rho^* \mathbf{E} + \mathbf{j} \times \mathbf{B}.$$

$\mathbf{F_e}$ is defined by Priest (1984) as the *Lorentz Force*. The fields $\mathbf{E}$ and $\mathbf{B}$ are defined as the *electric field* and the *magnetic field* respectively; they are measured in the SI units of newtons per coulomb ($\mathrm{N} \cdot \mathrm{c}^{-1}$) and teslas ($\mathrm{T} = \mathrm{N} \cdot \mathrm{A}^{-1} \cdot \mathrm{m}^{-1}$). Here, we highlight that the electric field acts upon the plasma even if it is at rest; $\rho^* \mathbf{E}$ can be thought of as an external electric force. We also see that electric charges give rise to a magnetic force $\mathbf{j}\times\mathbf{E}$ even in the absence of an electric field. This is the principle of electromagnetics.

The reader may notice that this new force has introduced seven new variables into our equation of motion. We will later show how assuming non-relativistic velocities and perfect conductivity allows us to neglect the electric force, treating $\mathbf{E}$ as a secondary quantity, however, we still seek a further relation for $\mathbf{B}$ in order to analyse MHD behaviour any further. For this, we turn to the equations of Maxwell, Ohm and Ampère.

7.2 The Induction Equation

The Maxwell equations are a set of four principle equations describing and relating the magnetic and electric fields. Priest (1984) gives the four equations, with eliminated electric displacement term, to be

$$\nabla \times \mathbf{B} = \gamma \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{7.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \tag{7.3}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho^*}{\epsilon} \tag{7.4}$$

where γ is the *magnetic permeability* ($\mathrm{N}\cdot\mathrm{A}^{-2}$), c is the *speed of light* and ϵ is the *permittivity of free space* measured in $\mathrm{F}\cdot\mathrm{m}^{-1}$ where $\mathrm{F} = \mathrm{s}^4\cdot\mathrm{A}^2\cdot\mathrm{m}^{-2}\cdot\mathrm{kg}^{-1}$ is defined as a farad. We will approximate γ and ϵ by their values in a vacuum, $\gamma_0 = 4\pi\times 10^{-7}\,\mathrm{N}\cdot\mathrm{A}^{-2}$ and $\epsilon_0 \approx 8.854\times 10^{-12}\,\mathrm{F}\cdot\mathrm{m}^{-1}$. It therefore follows that $c^2 = (\gamma_0\epsilon_0)^{-1}$. The first equation describes how magnetic fields can be produced either by the presence of electric charges or by a non-steady electric field while the third and fourth dictate how time varying magnetic fields and electric charges give rise to electric fields. An assumption that there are no magnetic poles results in the second equation.

During this discussion, we will only consider non-relativistic velocities of plasma such that $v_0 \ll c$ where $v_0 = \frac{l_0}{t_0}$ is a typical plasma speed. We also define E_0 and B_0 as typical lengths of vectors $\mathbf{E}$ and $\mathbf{B}$. Equation (7.3) therefore implies

$$\frac{E_0}{l_0} \approx \frac{B_0}{t_0}$$

and by equation (7.1)

$$\frac{B_0}{l_0} \approx \gamma_0\mathbf{j} + \frac{1}{c^2}\frac{E_0}{t_0} \approx \gamma_0\mathbf{j} + \frac{l_0 B_0}{c^2 t_0^2} = \gamma_0\mathbf{j} + \frac{v_0^2}{c^2}\frac{B_0}{l_0}.$$

Our non-relativistic assumption therefore allows us to neglect the term $\frac{1}{c^2}\frac{\partial E}{\partial t}$ in equation (7.1) to give

$$\nabla \times \mathbf{B} = \gamma_0 \mathbf{j} \tag{7.5}$$

which is known as *Ampère's law.* We also make use of *Ohm's law* which states that the current density is proportional to the total electric field given by

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7.6}$$

where σ is the *electric conductivity.* Rearranging for $\mathbf{E}$ and substituting into (7.3) we see

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{B}\right)$$

$$\Rightarrow \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left(\frac{\mathbf{j}}{\sigma}\right) + \nabla \times (\mathbf{v} \times \mathbf{B})$$

and, using Ampère's law, this becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

where $\eta = \frac{1}{\gamma_0 \sigma}$ is defined as the *magnetic diffusivity.* Expanding the double cross product we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \left[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\right]$$

but by (7.2) the divergence of the magnetic field is zero giving

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{7.7}$$

which is known as the *induction equation* and directly relates the velocity field to the magnetic field.

7.3 Ideal MHD Equation of Motion

In conjunction with non-relativistic velocity, one further assumption we will make in order to simplify our system is that of perfect conductivity. This assumes that a plasma has negligible electrical resistance, and therefore an electric field $\mathbf{E}'$ in a comoving reference frame vanishes, that is,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0.$$

It is these two assumptions that form the basis of ideal MHD. We now compare the magnitudes of the electric and magnetic terms of the Lorentz force to reveal an astounding consequence. Beginning with the electrical force, we see

$$\rho^*\mathbf{E} \sim \rho^* \frac{B_0 l_0}{t_0} \sim \rho^* B_0 v_0.$$

Now, by equation (7.4) and through imposing our perfect conductivity assumption, we find

$$\rho^* = \epsilon \nabla \cdot \mathbf{E} = -\frac{1}{c^2\gamma_0} \nabla \cdot (\mathbf{v} \times \mathbf{B})$$

implying that $\rho^* \sim \frac{1}{c^2\gamma_0} \frac{B_0 v_0}{l_0}$. We thus find

$$\rho^*\mathbf{E} \sim \frac{B_0 v_0}{c^2\gamma_0} \frac{B_0 v_0}{l_0} = \frac{v_0^2}{c^2} \frac{B_0^2}{l_0\gamma_0}.$$

Continuing with the magnetic force, Ampère's law implies that

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\gamma_0} \nabla \mathbf{B} \times \mathbf{B}$$

and moreover

$$\mathbf{j} \times \mathbf{B} \sim \frac{B_0^2}{\gamma_0 l_0}.$$

It there follows by our non-relativistic assumption that

$$\rho^*\mathbf{E} \ll \mathbf{j} \times \mathbf{B}$$

and we can neglect the electric term in the Lorentz force. Introducing the magnetic force into our equation of motion from fluid dynamics (5.14), we find

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p - \rho \nabla \varphi + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right) + \mathbf{j} \times \mathbf{B}$$

or, using Amperè's law and vector calculus identies to remove $\mathbf{j}$,

$$\begin{aligned}\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla \left(p + \frac{\mathbf{B}^2}{2\gamma_0} \right) - \rho \nabla \varphi \\ + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right) + \left(\frac{\mathbf{B}}{\gamma_0} \cdot \nabla \right) \mathbf{B},\end{aligned} \tag{7.8}$$

which we define as the *MHD equation of motion.*

7.4 Summary of Ideal MHD Equations

To summarise our equations for ideal MHD and to make full use of our approximation for γ, we diverge away from the standard units of metre and kilogram for distance and mass to centimetre (cm) and gram (g), which allows us to write $\gamma_0 = 4\pi$. Therefore, our system of MHD equations becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{7.7}$$

$$\begin{aligned} \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla \left(p + \frac{\mathbf{B}^2}{8\pi} \right) - \rho \nabla \varphi \\ + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right) + \left(\frac{\mathbf{B}}{4\pi} \cdot \nabla \right) \mathbf{B} \end{aligned} \tag{7.9}$$

to which we add the equation of mass conservation from fluid dynamics which is still applicable

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5.6}$$

Our system is not yet closed however; we still require a further relation for p. For this, we take the adiabatic equation of state from thermodynamics given by Gurnett and Bhattacharjee (2005) as

$$\frac{\partial}{\partial t} \left(p \rho^{-\frac{5}{3}} \right) = 0. \tag{7.10}$$

This assumes that heat cannot be exchanged between the plasma and its surroundings and closes our system of equations.

8 Magnetised Accretion Disc

Let us now reintroduce our steady-state disc from section 6.5. The disc is thin with Keplerian angular velocity $\Omega(r)$ to a first approximation. We are interested in how much the introduction of a magnetic field alters this steady state and therefore turn to a reference frame in Keplerian orbit. Following Balbus and Hawley (1998), let us define the fluctuation velocity to be $\mathbf{u}$ such that

$$u_r = v_r, \qquad u_\theta = v_\theta - r\Omega, \qquad u_z = v_z.$$

We substitute our newly defined velocities into the equation of motion in an expanded form, taking advantage of the fact that the newly defined velocity field is almost incompressible, allowing us to neglect the viscous term involving $\nabla(\nabla\cdot\mathbf{v})$ in comparison with ∇^2. This simplification is known as the *Boussinesq approximation.* The r-component is therefore given by

$$\rho\left[\frac{\partial u_r}{\partial t}+ u_r\frac{\partial u_r}{\partial r}+\frac{1}{r}(u_\theta+r\Omega)\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{1}{r}(u_\theta+r\Omega)^2\right]$$
$$=-\frac{\partial}{\partial r}\left(p+\frac{B^2}{8\pi}\right)-\rho\frac{\partial\varphi}{\partial r}+\left(\mathbf{B}4\pi\cdot\nabla\right)B_r$$
$$-\frac{B_\theta^2}{4\pi r}+\eta\left[\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial}{\partial\theta}(u_\theta+r\Omega)\right].$$

Our aim is to analyse the stability of the magnetised accretion disc, to which we restrict ourselves to cases over a small patch of the disc where fluctuation velocities are much less than the steady state velocities giving $u\ll r\Omega$. This local approximation allows us to neglect any curvature terms, i.e. terms involving reciprocals of r, reducing the equation to

$$\rho\left[\frac{\partial u_r}{\partial t}+ u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+\Omega\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-2\Omega u_\theta-r\Omega^2\right]$$
$$=-\frac{\partial}{\partial r}\left(p+\frac{\mathbf{B}^2}{8\pi}\right)-\rho\frac{\partial\varphi}{\partial r}+\left(\frac{\mathbf{B}}{4\pi}\cdot\right)B-r+\eta\nabla^2 u_r.$$

We have kept the θ-differential of u_r so this can be simplified using vector calculus operators. We also see from equation (6.24) that $\frac{\partial\varphi}{\partial r}\approx\frac{GM}{r^2}=r\Omega^2$, hence the centripetal force on the left cancels with the gravitational force on the right which we would expect. The r-component therefore becomes

$$\rho\left[\frac{\partial u_r}{\partial t}+(\mathbf{u}\cdot\nabla)u_r+\Omega\frac{\partial u_r}{\partial\theta}-2\Omega u_\theta\right]$$
$$=-\frac{\partial}{\partial r}\left(p+\frac{\mathbf{B}^2}{8\pi}\right)+\left(\frac{\mathbf{B}}{4\pi}\cdot\right)B-r+\eta\nabla^2 u_r$$

or, redefining $\frac{D}{Dt}$ as $\frac{D}{Dt}=\frac{\partial}{\partial t}+(\mathbf{u}\cdot\nabla)+\Omega\frac{\partial}{\partial\theta}$,

$$\rho\left(\frac{Du_r}{Dt}-2\Omega u_\theta\right)=-\frac{\partial}{\partial r}\left(p+\frac{\mathbf{B}^2}{8\pi}\right)+\left(\frac{\mathbf{B}}{4\pi}\cdot\right)B-r+\eta\nabla^2 u_r. \tag{8.1}$$

Continuing on to the θ and z components with a similar approach, we find

$$\rho\left(\frac{Du_\theta}{Dt}+\frac{\kappa^2}{2\Omega}u_r\right)=-\frac{1}{r}\frac{\partial}{\partial\theta}\left(p+\frac{\mathbf{B}^2}{8\pi}\right)+\left(\frac{\mathbf{B}}{4\pi}\cdot\nabla\right)B_r+\eta\nabla^2 u_\theta, \tag{8.2}$$

$$\rho\frac{Du_z}{Dt}=-\frac{\partial}{\partial z}\left(p+\frac{\mathbf{B}^2}{8\pi}\right)-\rho\frac{\partial\varphi}{\partial z}+\left(\frac{\mathbf{B}}{4\pi}\cdot\nabla\right)B_z+\eta\nabla^2 u_z \tag{8.3}$$

where $\kappa^2=\frac{1}{r^3}\frac{d(r^4\Omega^2)}{dr}$ is defined as the *epicyclic frequency.* In the case of Keplerian rotation $\kappa^2=\Omega^2$. We apply the same substitution and approximations to the induction equations which can be rearranged to give

$$\frac{\partial\mathbf{B}}{\partial t}=-\mathbf{B}(\nabla\cdot\mathbf{v})+(\mathbf{B}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{B}+\eta\nabla^2\mathbf{B}$$

using Maxwell's equations to equate the divergence of the magnetic field to zero. Introducing our expression for $\mathbf{v}$, the r-component is given by

$$\begin{aligned}\frac{\partial B_r}{\partial t} = -B_r\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta + r\Omega) + \frac{\partial u_z}{\partial z}\right] \\ + \left[B_r\frac{\partial u_r}{\partial r} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + B_z\frac{\partial u_r}{\partial z} - \frac{B_\theta(u_\theta + r\Omega)}{r}\right] \\ - \left[u_r\frac{\partial B_r}{\partial r} + \frac{u_\theta + r\Omega}{r}\frac{\partial B_r}{\partial z} + u_z\frac{\partial B_r}{\partial z} + \frac{B_\theta(u_\theta + r\Omega)}{r}\right] \\ + \eta\left(\nabla^2 B_r - \frac{B_r}{r^2} - \frac{2}{r^2}\frac{\partial B_\theta}{\partial \theta}\right).\end{aligned}$$

Omitting the reciprocals of r, this reduces to

$$\begin{aligned}\frac{\partial B_r}{\partial t} =& - B_r\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + \left(B_r\frac{\partial u_r}{\partial r} + \frac{B_\theta}{r}\frac{\partial u_r}{\partial \theta} + B_z\frac{\partial u_r}{\partial z}\right) \\ & - \left(v_r\frac{\partial B_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial B_r}{\partial \theta} + u_z\frac{\partial B_r}{\partial z}\right) - \Omega\frac{\partial B_r}{\partial \theta} + \eta\nabla^2 B_r\end{aligned}$$

which can be written

$$\frac{\partial B_r}{\partial t} + (\mathbf{u}\cdot\nabla)B_r + \Omega\frac{\partial B_r}{\partial \theta} = -B_r(\nabla\cdot\mathbf{u}) + (\mathbf{B}\cdot\nabla u_r + \eta\nabla^2 B_r). \tag{8.4}$$

Again, following in a similar way, the θ and z-components become

$$\frac{DB_\theta}{Dt} - B_r\frac{d(r\Omega)}{dr} = -B_\theta(\nabla\cdot\mathbf{u}) + (\mathbf{B}\cdot\nabla)u_\theta + \eta\nabla^2 B_\theta, \tag{8.5}$$

$$\frac{DB_z}{Dt} = -B_z(\nabla\cdot\mathbf{u}) + (\mathbf{B}\cdot\nabla)u_z + \eta\nabla^2 B_z. \tag{8.6}$$

Given this reduced system of equations describing the behaviour of the magnetised accretion disc, we continue by performing a linear perturbation analysis to gain an understanding of how the disc may become unstable and allow a sufficient mechanism for accretion.

8.1 Linear Perturbation Analysis

We pertube the steady plasma by small local linear disturbances such that

$$\mathbf{u} = \mathbf{u}', \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}', \qquad p = p_0 + p', \qquad \rho = \rho_0 + \rho',$$

where the disturbance terms, marked with dashes, are of the form $a' = \Re\left[\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$ such that $\hat{a}$ are linear amplitudes, $\mathbf{k}$ is the wave vector, $\mathbf{r}$ the position vector and ω the angular frequency. Following Balbus and Hawley (1998), we will only consider magnetic fields with azimuthal and vertical components, suggesting $B_r = 0$, since this does not affect our final result a substantial amount and simplifies our problem extensively. For a similar reason, we consider only wave numbers in the z-component giving $\mathbf{k} = k\mathbf{e_z}$. We begin with the conservation of mass equation:

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + \nabla[(\rho_0 + \rho')\mathbf{v}'] = 0 \Rightarrow \frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}' + \rho' \mathbf{v}') = 0.$$

Since we assumed disturbances to be small, we can linearise so there are no second order disturbance quantities, allowing us to omit the $\rho'\mathbf{v}'$ term. This local, linear approach is known as a WKB (Wentzel, Kramers and Brillouin) stability analysis. Substituting in our perturbation expression we find

$$\frac{\partial}{\partial t}\Re\left[\hat{\rho}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] + \nabla \cdot \rho_0 \Re\left[\hat{\mathbf{v}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = 0$$

$$\Rightarrow \Re\left[\hat{\rho}\frac{\partial}{\partial t}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] + \Re\left[\rho_0 \nabla \cdot \hat{\mathbf{v}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = 0$$

$$\Rightarrow -\hat{\rho}\omega + \rho_0 v_z k = 0,$$

therefore, dropping the zero subscript, the conservation of mass equation gives us the linear relation

1. $-\omega\frac{\hat{\rho}}{\rho} + ku_z = 0.$ (from 5.6)

We continue a WKB analysis on the components of the equation of motion and the induction equation, together with the equation of state to obtain the following linear relations:

2. $-i\omega\hat{u}_r - 2\Omega\hat{u}_\theta - i\frac{kB_z}{4\pi\rho}\hat{B}_r = 0;$ (from 8.1)
3. $-i\omega\hat{u}_\theta + \frac{\kappa^2}{2\Omega}\hat{u}_r - i\frac{kB_z}{4\pi\rho}\hat{B}_\theta = 0;$ (from 8.2)
4. $-\omega\hat{u}_z + k\left(\frac{\hat{P}}{\rho} + \frac{B_\theta\hat{B}_\theta}{4\pi\rho}\right) = 0;$ (from 8.3)
5. $-\omega\hat{B}_r = kB_z\hat{u}_r;$ (from 8.4)
6. $-i\omega\hat{B}_\theta = \hat{B}_r\frac{d\Omega}{d\ln r} + ikB - z\hat{u}_\theta - B_\theta ik\hat{u}_z;$ (from 8.5)
7. $\hat{B}_z = 0;$ (from 8.6)
8. $\frac{\hat{P}}{P} = \frac{5}{3}\frac{\hat{\rho}}{\rho};$ (from 7.10)

We have used the fact that $r\frac{d\Omega}{dr} = \frac{d\Omega}{d\ln r}$. This gives us a total of eight linear equations for eight unknowns which can be reduced down to the following three equations for $\hat{B}_r$, $\hat{B}_\theta$ and $\hat{P}$:

$$\left(\omega^3 + \frac{\omega k^2 B_z^2}{4\pi\rho} - \kappa^2\omega\right)\hat{B}_r - \frac{ik^2B_z^2\Omega}{2\pi\rho}\hat{B}_\theta = 0;$$

$$\left(\frac{k}{\rho} - \frac{\omega^2}{ka^2\rho}\right)\hat{P} + \left(\frac{kB_\theta}{4\pi\rho}\right)\hat{B}_\theta = 0;$$

$$\left(\frac{d\Omega}{d\ln r} - \frac{\omega^2}{2\Omega} + \frac{k^2B_z^2}{8\pi\omega\rho}\right)\hat{B}_r + i\omega\hat{B}_\theta - \frac{B_\theta i\omega}{a^2\rho}\hat{P} = 0.$$

The reduced system of equations has non-trivial solutions if the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} \omega^3 + \frac{\omega k^2 B_z^2}{4\pi\rho} - \kappa^2\omega & -\frac{ik^2B_z^2\Omega}{2\pi\rho} & 0 \\ 0 & \frac{ik^2B_z^2\Omega}{2\pi\rho} & \frac{k}{\rho} - \frac{\omega^2}{ka^2\rho} \\ \frac{d\Omega}{d\ln r} - \frac{\omega^2}{2\Omega} + \frac{k^2B_z^2}{8\pi\omega\rho} & i\omega & -\frac{B_\theta i\omega}{a^2\rho} \end{pmatrix}$$

is zero. We find

$$\begin{aligned}|\mathbf{A}| =& \frac{i}{ka^2\rho}\left(\omega^3 - \frac{\omega k^2 B_z^2}{4\pi\rho} - \kappa^2\omega\right)\left(-\frac{k^2B_\theta^2\omega}{4\pi\rho} - k^2a^2\omega + \omega^3\right) \\ &- \frac{ikB_z^2}{4\pi\rho^2a^2}\left(2\Omega\frac{d\Omega}{d\ln r} - \omega^2 + \frac{k^2}{B_z^2}4\pi\rho\right)\left(k^2a^2 - \omega^2\right),\end{aligned}$$

therefore, defining $\mathbf{u}_A = \frac{\mathbf{B}}{4\pi\rho}$ as the *Alfvén velocity*, non-trivial solutions to our system of linearised perturbation equations exist given that

$$\begin{aligned}&[\omega^3 - \omega(\mathbf{k}\cdot\mathbf{u}_A]^2 - \kappa^2\omega)(-k^2u_{A\theta}^2\omega - k^2a^2\omega + \omega^3) \\ &\quad - (\mathbf{k}\cdot\mathbf{u}_A)^2\left[\frac{d\Omega^2}{d\ln r} - \omega^2 + (\mathbf{k}\cdot\mathbf{u}_A)^2\right](k^2a^2 - \omega^2) = 0.\end{aligned} \tag{8.7}$$

This relation between the wave frequency and wave number is defined as a *dispersion relation* and gives an insight into the behaviour of wave like solutions. Equation (8.7) be separated into terms with and without rotation effects to give

$$\begin{aligned}&[\omega^2 - (\mathbf{k}\cdot\mathbf{u}_A)^2][\omega^4 - k^2\omega^2(a^2 + \mathbf{u}_A^2) + k^2a^2(\mathbf{k}\cdot\mathbf{u}_A)^2] \\ &- \left\{\kappa^2\omega^4 - \omega^2\left[\kappa^2k^2(a^2 + u_{A\theta}^2) + (\mathbf{k}\cdot\mathbf{u}_A)^2\frac{d\Omega^2}{d\ln r}\right]\right\} - k^2a^2(\mathbf{k}\cdot\mathbf{u}_A)^2\frac{d\Omega^2}{d\ln r} = 0.\end{aligned} \tag{8.8}$$

This cubic equation in ω^2 can be solved in order to gain a specific expression for ω in terms of k and Ω, which will allow us to analyse the MHD waves caused by small disturbances to the accretion disc.

8.2 MHD Waves and the Origin of Instability

To firstly introduce the waves of MHD, we consider the non-rotating situation where $\Omega = 0 \Rightarrow \kappa^2 = 0$, which gives the dispersion relation

$$[\omega^2 - (\mathbf{k}\cdot\mathbf{u}_A)^2][\omega^4 - k^2\omega^2(a^2 + \mathbf{u}_A^2) + k^2a^2(\mathbf{k}\cdot\mathbf{u}_A)^2] = 0.$$

One solution is clearly the case of

$$\omega^2 = \omega_A^2 = (\mathbf{k}\cdot\mathbf{u}_A)^2 = \frac{1}{4\pi\rho}\mathbf{k}^2\mathbf{B}^2\cos^2\theta_B = k^2u_A^2\cos^2\theta_B$$

where θ_B is the angle that the wave propagation makes with the magnetic field. Such waves are known as *Alfvén waves* in recognition of the physicist Hannes Alfvén (1908-1995) who first described this class of MHD waves. The *phase speed* of a wave is given by the scalar $c_p = \frac{\omega}{k}$; Alfvén waves therefore have phase speeds

$$c_{P_A} = \frac{\omega_A}{k} = u_A\cos\theta_B$$

where we take the positive square root so they take the same direction as the magnetic field. We see they are fastest along the lines of the magnetic field but stationary in the normal direction.

We are then left with the quadratic relation

$$\omega^4 - k^2\omega^2(a^2+\mathbf{u}_A^2) + k^2a^2(\mathbf{k}\cdot\mathbf{u}_A)^2 = 0$$

for ω^2 which has solutions

$$\omega_\pm^2 = k^2\frac{1}{2}\left[a^2+u_A^2 \pm \sqrt{(a^2+u_A^2)^2 - 4u_A^2a^2\cos^2\theta_B}\right].$$

These waves correspond to the fast and slow *magnetoacoustic waves*; the slow mode corresponding to the negative sign solution and the fast to the positive. The naming convention arises since the phase speed of Alfvén waves lies between that of the slow and fast magnetoacoustic waves. The fast mode may be thought of as a sound wave distorted by a magnetic field while the slow mode represents the opposing magnetic tension and pressure. If the magnetic field is weak ($u_A \to 0$), the slow mode is reduced to an Alfvén wave while the fast mode becomes a sound wave suggesting that in this limit, the slow and Alfvén waves are closely related. Without rotation, there seems no reason to suggest that the plasma would become unstable.

We continue onto analysing the dispersion relation in the case where there is rotation present. Fixing the values $(\mathbf{k}\cdot\mathbf{u_A})^2 = 1$, $ku_{a\theta} = 2$ and $ka = 5$, such that all frequencies are in units of $\mathbf{k}\cdot\mathbf{u_A}$, equation (8.8) reduces to a cubic for ω^2 and Ω^2 given by

$$\omega^6 - \omega^4(31+\Omega^2) + (55+26\Omega^2)\omega^2 + (75\Omega^2 - 25) = 0.$$

This can be solved to obtain three expressions for ω^2 in terms of Ω^2 using a number of different approaches (such as Cardano's method, Vieta's substitution or Lagrange's method) corresponding to the three types of waves previously discussed. A plot of the three solutions is given in figure 8.

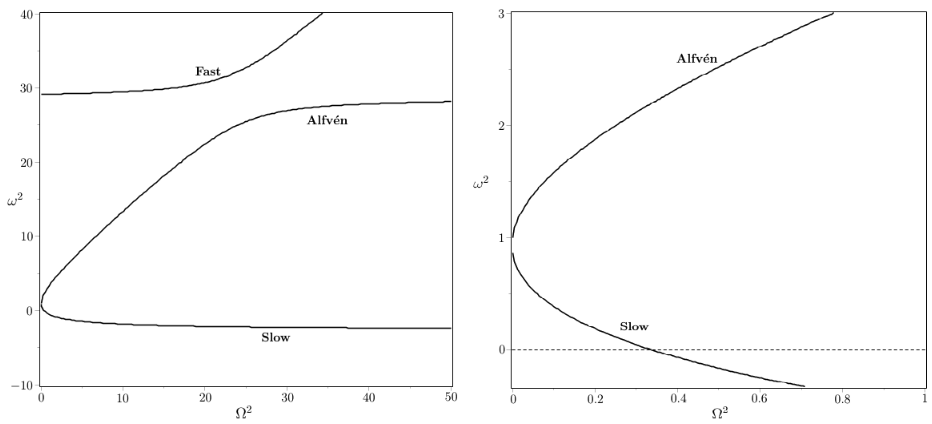

Figure 8: The three solutions to the rotating dispersion relation, corresponding to the three types of MHD waves. Adapted from Balbus and Hawley (1998).

An astounding result can be seen within the solution for the slow magnetoacoustic waves as Ω^2 increases; ω^2 becomes negative. This implies that the wave frequency, ω, becomes complex, which, by the nature of our WKB perturbations, would imply exponential growth in our small perturbations and provide an origin of instability as opposed to a well-behaving wave. An electrically conducting accretion disc has the potential to induce turbulence, and consequently introduce an efficient mechanism for the transport of angular momentum.

8.3 Discussion of Linear Stability Analysis

Our linear perturbation analysis has shown that, given sufficient angular velocity, the inclusion of a magnetic field within an accretion disc has the potential to induce an instability. Balbus and Hawley (1998) shown that the forces associated with a linear perturbation to a rotating disc obey the same equations as two orbiting point particles connected by a massless spring. This analogy can therefore be used to discuss the instability and give a physical insight into its origin and behaviour.

Let us consider two point particles, m_1 and m_2, connected by a massless spring in the same Keplerian orbit around a central mass. Now suppose their positions are displaced slightly, with m_1 adopting an orbit slightly closer to the central mass than m_2. The particle with mass m_1 therefore assumes a higher angular velocity than m_2. As m_1 orbits on this faster rate, the distance between the two particles grows and the tension in the spring connecting them increases, pulling m_2 forwards and m_1 backwards, which consequently transports angular momentum outwards. The distance between the two particles is further increased as the change in angular momenta causes m_1 to take an even smaller orbit and m_2 to drift further away. Tension in the spring is continuously increased further and the process becomes turbulent. This is the basis of the *magnetorotational instability* (MRI). It can be shown that this mechanism has the capability to produce the luminosities observed in accretion discs. We must highlight that this instability is based on the existence of a *weak* magnetic field; a strong magnetic field will invalidate this theory since the restoring force will stabilise any small perturbations in a similar way to a strong spring.

Balbus and Hawley (1998) show that the criterion for stability within the accretion disc becomes $\frac{d\Omega^2}{d\ln r} > 0$, or $r\frac{d\Omega}{dr} > 0$; an electrically conducting accretion disc is therefore unstable if angular velocity decreases outwards. We have shown that for a Keplerian disc this is always the case. In a generalisation of this analysis, it can be shown that this instability is evident in discs without Keplerian orbit where the thin approximation breaks down. It is remarkable that such an efficient instability beautifully evolves as a direct consequence of the inclusion of a magnetic field.

9 Conclusion of the Dynamics of Accretion Discs

Our analysis of the dynamics of accretion discs has taken us on a journey through the fundamental states of matter within our universe: solid particles; fluids and plasma.

An application of the simple laws from classical mechanics lead us to conclude that particles in the presence of a dominating gravitational field would adopt coplanar orbits, following circular motion in equilibrium. In order for further energy to be extracted from the system, it became evident that a process had to be in place which would influence an outward transportation of angular momentum and an inward transportation of mass, for which we turned to the viscous stresses described in fluid dynamics.

A continuous disc in differential rotation had the potential to meet our conditions for energy dissipation through shear stress. Manipulation of the conservation of mass equation and the Navier-Stokes equation of motion highlighted that shear stress caused inner parts of the disc to transfer angular momentum outwards, consequently motivating the mass of the inner parts to take smaller orbits. Through analysing further the case where the disc was steady, scaling arguments lead us to believe that shear stress arising from molecular viscosity was not the primary source of dissipation, yielding enormous Reynolds numbers and excruciatingly slow accretion rates. Our discussion of the Rayleigh stability criterion, which could not be broken under the subject of fluid dynamics, influenced us to consider electrically conducting accretion discs.

The equations of a non-relativistic, perfectly conduction plasma allowed us to deduce the MHD equation of motion and the induction equation, which we were able to apply to a steady disc in Keplerian orbit. A simple WKB linear stability analysis reduced these equations to a dispersion relation, relating the frequency of wave like solutions to their wave number, which could be simplified to show the three waves of MHD in a non-rotating limit. Upon the reintroduction of rotation into this system, our dispersion relation extraordinarily highlighted that under sufficient angular velocity, the wave frequency associated with a small perturbation was complex, causing the disturbances to grow exponentially. The comparison by Balbus and Hawley (1998) of this instability under a weak magnetic field to two point particles connected by a massless spring highlighted its mechanics, which can be attributed as a candidate for the origin of dissipation within accretion discs, known as the magnetorotational instability.

To this day, the role of MRI not only remains an active area of research within accretion discs, but also other areas of plasma dynamics. It plays a significant part in the study of the Taylor-Couette flow, consisting of a viscous fluid contained in the space between two rotating cylinder, unbounded in the z-directions. MRI is also though to occur in a variety of other astrophysical systems such as planetary dynamos and internal rotation within stars. Our analysis into the dynamics of accretion discs has lead us to an area of plasma physics that is, and will remain, a fundamental interest to astrophysicists.

Appendix

A Expressions in Cylindrical Coordinates

Let f be a scalar quantity and $\mathbf{F}$ be a vector. The vector calculus operators can be expanded in cylindrical coordinates as follows:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{e_z}; \tag{A.1}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}; \tag{A.2}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}; \tag{A.3}$$

$$\begin{aligned}\nabla^2 \mathbf{F} = &\left(\nabla F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta}\right)\mathbf{e_r} \\ &+ \left(\nabla F_\theta - \frac{F_\theta}{r^2} + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta}\right)\mathbf{e}_{\boldsymbol{\theta}} + (\nabla F_z)\mathbf{e_z};\end{aligned} \tag{A.4}$$

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{pmatrix} \mathbf{e_r} & r\mathbf{e}_{\boldsymbol{\theta}} & \mathbf{e_z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{pmatrix}; \tag{A.5}$$

$$\begin{aligned}\frac{Df}{Dt} &= \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla)f \\ &= \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + v_z\frac{\partial f}{\partial z};\end{aligned} \tag{A.6}$$

$$\begin{aligned}\frac{D\mathbf{F}}{Dt} &= \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{F} \\ &= \frac{\partial \mathbf{F}}{\partial t} + \left(v_r\frac{\partial F_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial F_r}{\partial \theta} + v_z\frac{\partial F_r}{\partial z} - \frac{v_\theta F_\theta}{r}\right)\mathbf{e_r} \\ &\quad + \left(v_r\frac{\partial F_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial F_\theta}{\partial \theta} + v_z\frac{\partial F_\theta}{\partial \theta} + \frac{v_\theta F_r}{r}\right)\mathbf{e}_{\boldsymbol{\theta}} \\ &\quad + \left(v_r\frac{\partial F_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial F_z}{\partial \theta} + u_z\frac{\partial F_z}{\partial z}\right)\mathbf{e_z};\end{aligned} \tag{A.7}$$

$$\nabla\mathbf{F} = \begin{pmatrix} \frac{\partial F_r}{\partial r} & \frac{1}{r}\left(\frac{\partial F_r}{\partial \theta} - F_\theta\right) & \frac{\partial F_r}{\partial z} \\ \frac{\partial F_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial F_\theta}{\partial \theta} + F_r\right) & \frac{\partial F_\theta}{\partial z} \\ \frac{\partial F_z}{\partial r} & \frac{1}{r}\frac{\partial F_z}{\partial \theta} & \frac{\partial F_z}{\partial z} \end{pmatrix}. \tag{A.8}$$

B Vector Calculus Identites

Let f be a scalar and $\mathbf{F}$ and $\mathbf{G}$ be vectors. The following vector calculus identities hold true:

$$\nabla \times \nabla f = 0; \tag{B.1}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0; \tag{B.2}$$

$$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f; \tag{B.3}$$

$$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}; \tag{B.4}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} - \mathbf{G}(\nabla \cdot \mathbf{F}) + \mathbf{F}(\nabla \cdot \mathbf{G}); \tag{B.5}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}; \tag{B.6}$$

$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}); \tag{B.7}$$

$$(\nabla \times \mathbf{F}) \times \mathbf{F} = (\mathbf{F} \cdot \nabla)\mathbf{F} - \nabla\left(\frac{1}{2}\mathbf{F}^2\right). \tag{B.8}$$

The divergence theorem states, for a volume V bounded by a simple closed surface S with outward normal $\mathbf{n}$,

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \mathbf{F}\, dV, \qquad \int_S f\mathbf{n}\, dS = \int_V \nabla f\, dV.$$

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